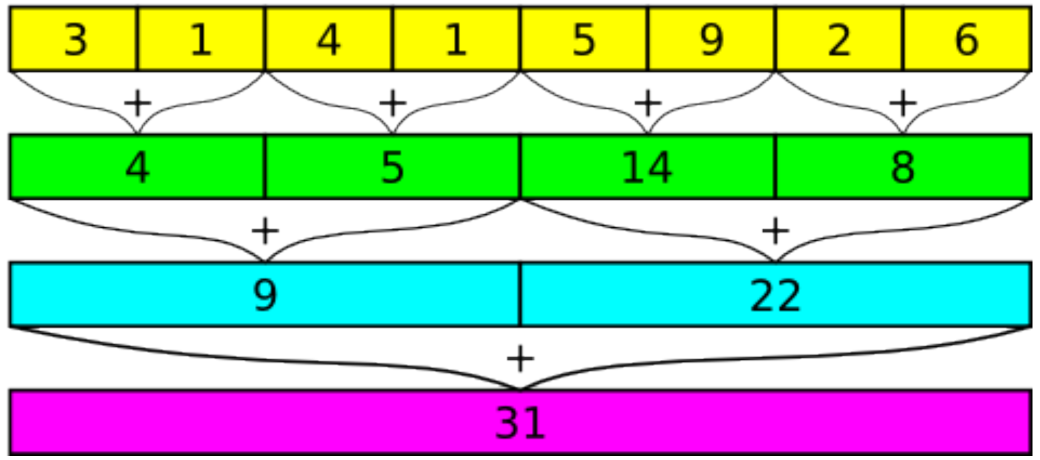
Massively Parallel Algorithms
Parallel Prefix Sum And Its Applications



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- Remember the reduction operation
  - Extremely important/frequent operation → Google's *MapReduce*
- Definition prefix sum:
  Given an input sequence $A = (a_0, a_1, a_2, \ldots, a_{n-1})$,
  the (inclusive) prefix sum of this sequence is the output sequence
  $$\hat{A} = (a_0, a_1 \oplus a_0, a_2 \oplus a_1 \oplus a_0, \ldots, a_{n-1} \oplus \cdots \oplus a_0)$$
  where $\oplus$ is an arbitrary binary associative operator.
- The exclusive prefix sum is
  $$\hat{A}' = (\iota, a_0, a_1 \oplus a_0, \ldots, a_{n-2} \oplus \cdots \oplus a_0)$$
  where ι is the identity/zero element, e.g., 0 for the + operator.
- The prefix sum operation is sometimes also called a scan (operation)

- Example:
  - Input: $A = (3\ 1\ 7\ 0\ 4\ 1\ 6\ 3)$
  - Inclusive prefix sum: $\hat{A} = (3\ 4\ 11\ 11\ 15\ 16\ 22\ 25)$
  - Exclusive prefix sum: $\hat{A}' = (0\ 3\ 4\ 11\ 11\ 15\ 16\ 22)$
- Further variant: backward scan
- Applications: many!
  - For example: polynomial evaluation (Horner's scheme)
  - In general: "What came before/after me?"
  - "Where do I start writing my data?"

- The prefix sum problem appears to be "inherently sequential"

Variation: Segmented Scan

- Input: segments of numbers in one large vector

3	1	7	0	4	1	6	3	← Payload data
1	0	1	0	0	1	0	0	← Flags array (*head-tail flags*, "1" = new segment)

- Task: scan (prefix-sum) within each segment
- Output: prefix-sums for each segment, in one vector

0	3	0	7	7	0	1	7

- Forms the basis for a wide variety of algorithms:
  - E.g., Quicksort, Sparse Matrix-Vector Multiply, Convex Hull
- Note: take care to store the flags array space- and bandwidth-efficient! (one integer per flag is very in-efficient)

Application from "Everyday" Life

- Given:
  - A 100-inch sandwich
  - 10 persons
  - We know how many inches each person wants: [3 5 2 7 10 4 3 0 8 1]
- Task: cut the sandwich quickly
- Sequential method: one cut after another (3 inches first, 5 inches next, ...)
- Parallel method:
  - Compute prefix sum
  - Make cuts in parallel with 10 knives
  - How quickly can we compute the prefix sum?

Illustration of the Importance of the Scan Operation

- Under the different parallel RAM (PRAM) models, the following graph algorithms have the given parallel complexities
- Assuming the scan operation is a primitive that has *unit time* costs, then the parallel complexities are reduced (or not) as follows:

	Model		
Algorithm	EREW	CRCW	Scan
Graph Algorithms (n vertices, m edges, m processors)			
Minimum Spanning Tree	$\lg^2 n$	$\lg n$	$\lg n$
Connected Components	$\lg^2 n$	$\lg n$	$\lg n$
Maximum Flow	$n^2 \lg n$	$n^2 \lg n$	n^2
Maximal Independent Set	$\lg^2 n$	$\lg^2 n$	$\lg n$
Biconnected Components	$\lg^2 n$	$\lg n$	$\lg n$
Sorting and Merging (n keys, n processors)			
Sorting	$\lg n$	$\lg n$	$\lg n$
Merging	$\lg n$	$\lg \lg n$	$\lg \lg n$
Computational Geometry (n points, n processors)			
Convex Hull	$\lg^2 n$	$\lg n$	$\lg n$
Building a K-D Tree	$\lg^2 n$	$\lg^2 n$	$\lg n$
Closest Pair in the Plane	$\lg^2 n$	$\lg n \lg \lg n$	$\lg n$
Line of Sight	$\lg n$	$\lg n$	1
Matrix Manipulation ($n \times n$ matrix, n^2 processors)			
Matrix $\times$ Matrix	n	n	n
Vector $\times$ Matrix	$\lg n$	$\lg n$	1
Matrix Inversion	$n \lg n$	$n \lg n$	n

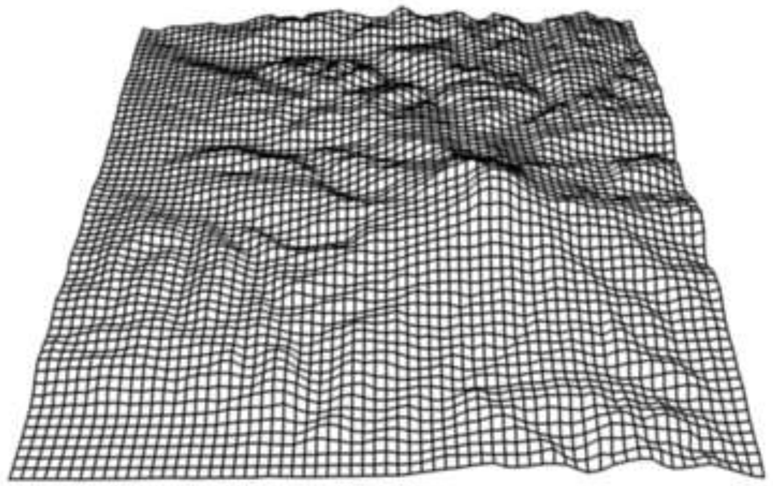
Guy E. Blelloch: Vector Models for Data-Parallel Computing

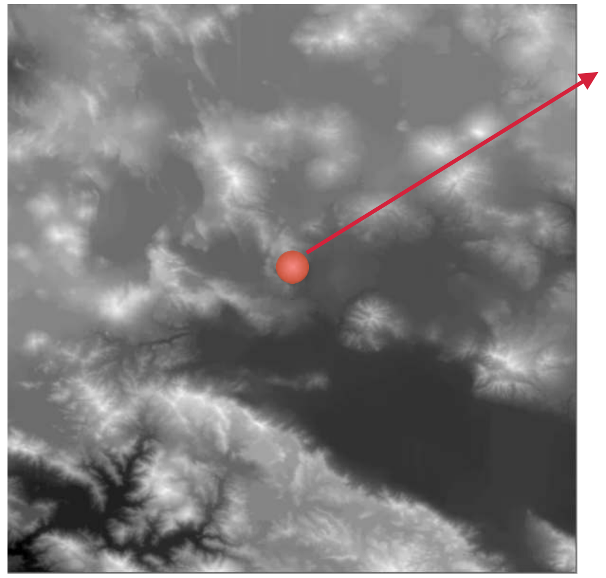
EREW = exclusive-read, exclusive-write PRAM
CRCW = concurrent-read, concurrent-write PRAM
Scan = EREW with scan as unit-cost primitive

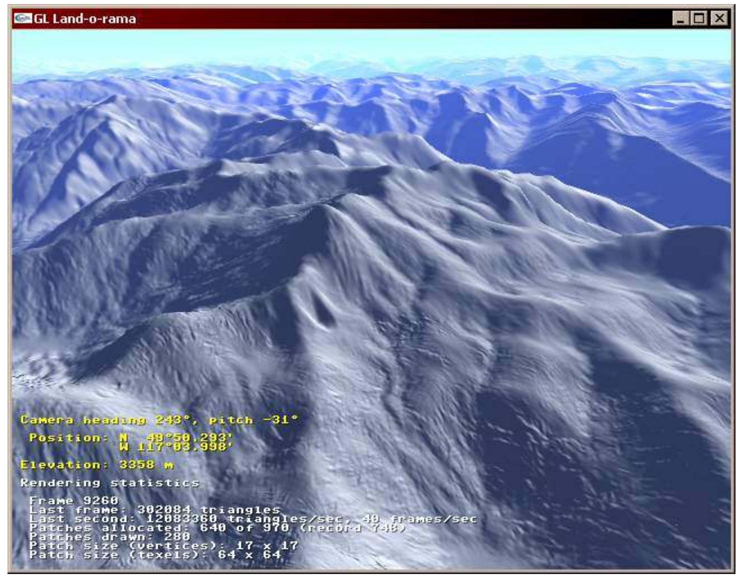
- Actually, *prefix-sum* (a.k.a. *scan)* was considered such an important operation, that it was implemented as a primitive in the *CM-2 Connection Machine* (of Thinking Machines Corp.)

Example: Line-of-Sight

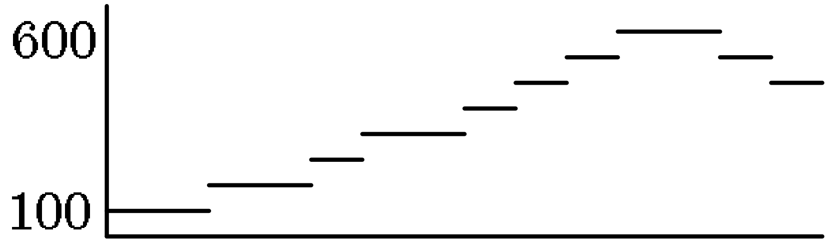
- Given:
  - Terrain as grid of height values (*height map*)
  - Point X in the grid (our "viewpoint", has a height, too)
  - Viewing direction, we can look up and down, but not to the left or right
- Problem: find all *visible* points in the grid along the viewing direction
- Assumption: we have already extracted a vector of heights from the grid containing all cells' heights that are along our viewing direction



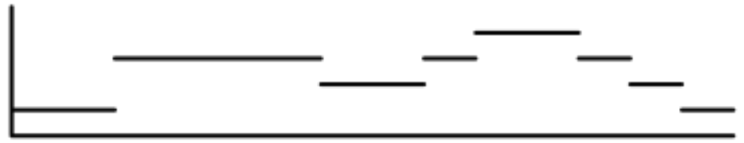




- The algorithm:
  - Convert height vector to vertical angles (as seen from X) → A
    - One thread per vector element
  - Perform max-scan on angle vector (i.e., prefix sum with the max operator) → $\hat{A}$
  - Test $\hat{a}_i < a_i$, if true then grid point is visible form X



Height vector

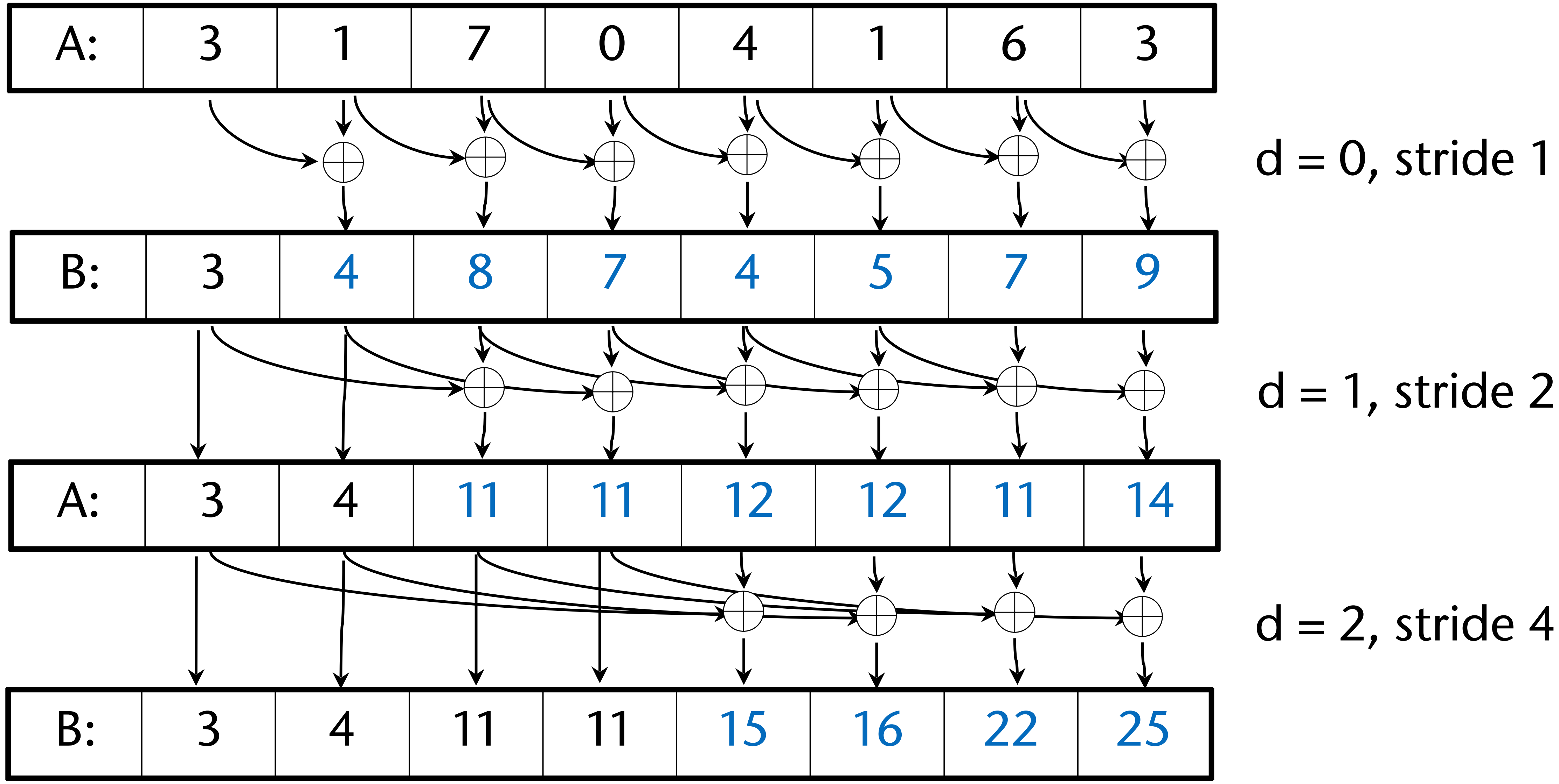


Angle vector (A)

Max-scan of angle vector ($\hat{A}$)

The Hillis-Steele Algorithm (MassPar Pattern)

- Iterate log(n) times:



- Notes:
  - Blue = active threads
  - Each thread reads from another lane, too → must use barrier sync
    - Could save one barrier by double buffering

- The algorithm as pseudo-code:

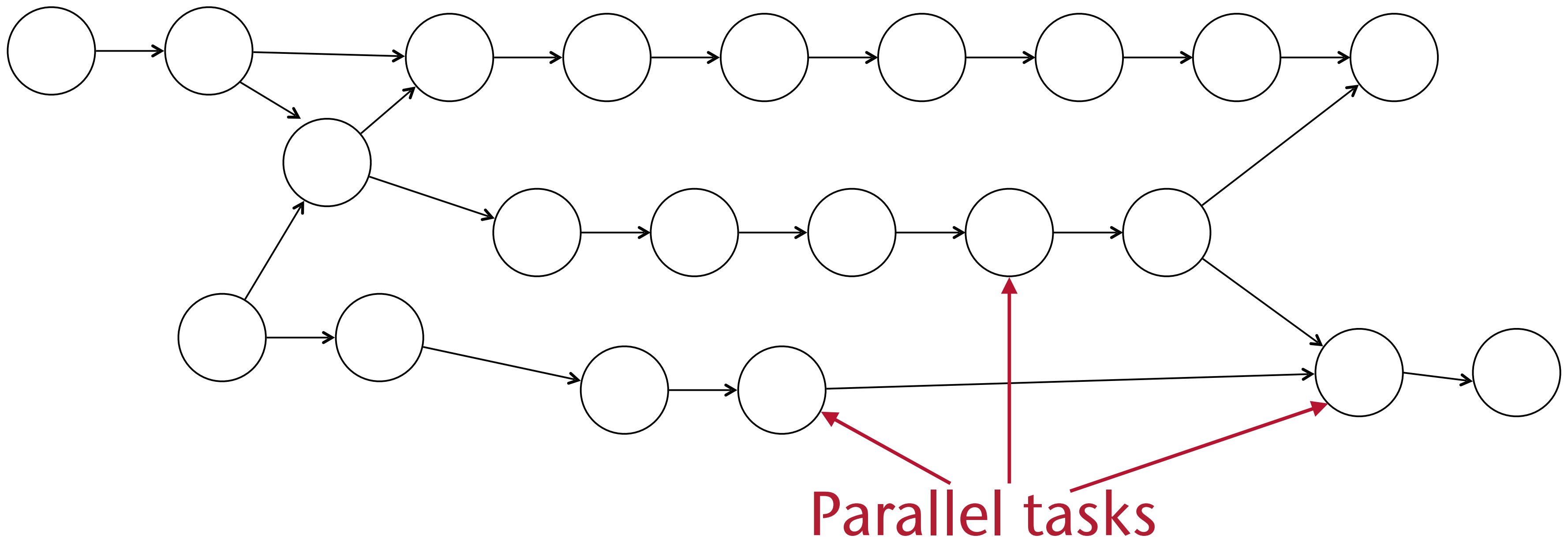
```
forall i in parallel do          // n threads
   for d = 0...log(n)-1:
      if i >= 2^d :
         x[i] = x[ i - 2^d ] + x[i]
```

  - Note: barrier synchronization omitted for clarity
- Remark: precision is usually better than the naïve sequential algo
  - Because, in the parallel version, summands (in each iteration) tend to be of the same order
- Algorithmic technique: recursive/iterative doubling technique = "Accesses or actions are governed by increasing powers of 2"
  - Remember the algo for maintaining dynamic arrays? (2nd/1st semester)

Definitions

- Depth complexity $D(n)$ = "#iterations" = parallel running time $T_p(n)$
  - (Think of the loops unrolled and "baked" into a hardware pipeline)
  - Sometimes also called step complexity
- Work complexity $W(n)$ = total number of operations performed by all threads
  - With *sequential* algorithms, *work complexity* = *time complexity*
- Work-efficient:
  A parallel algorithm is called *work-efficient*, if it performs *no more work* than the sequential one (in Big-O notation)

- Visual definition of depth/work complexity:
  - Express computation as a dependence graph of parallel tasks:



  - Work complexity = total amount of work performed by all tasks
  - Depth complexity = length of the "critical path" in the graph

- Parallel algorithms should be always both work and depth efficient!

Complexity of the Hillis-Steele Algorithm

- Depth $D(n) = T_p(n) = \#$ iterations $= \log(n)$ → good
- In iteration d: #additions $= n - 2^{d-1}$
- Total number of add operations = work complexity

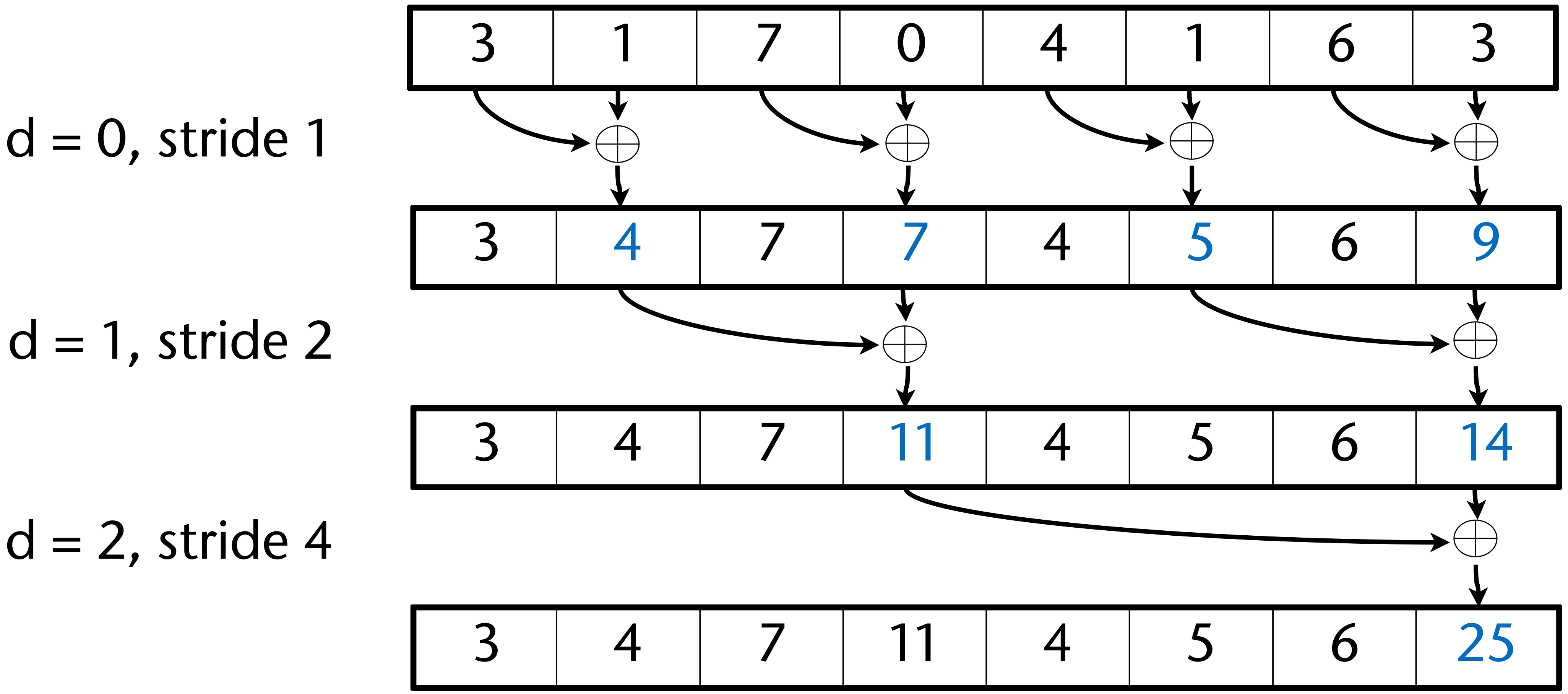
$$W(n) = \sum_{d=1}^{\log_2 n} (n - 2^{d-1}) = \sum_{d=1}^{\log_2 n} n - \sum_{d=1}^{\log_2 n} 2^{d-1} = n \cdot \log n - n \in O\left(n \log n\right)$$

- Conclusion: not work-efficient
  - A factor of $\log(n)$ can hurt: amounts to 20× for 10^6 elements

The Blelloch Algorithm (here for Exclusive Scan)

- Consists of two phases: *up-sweep* (= reduction) and *down-sweep*

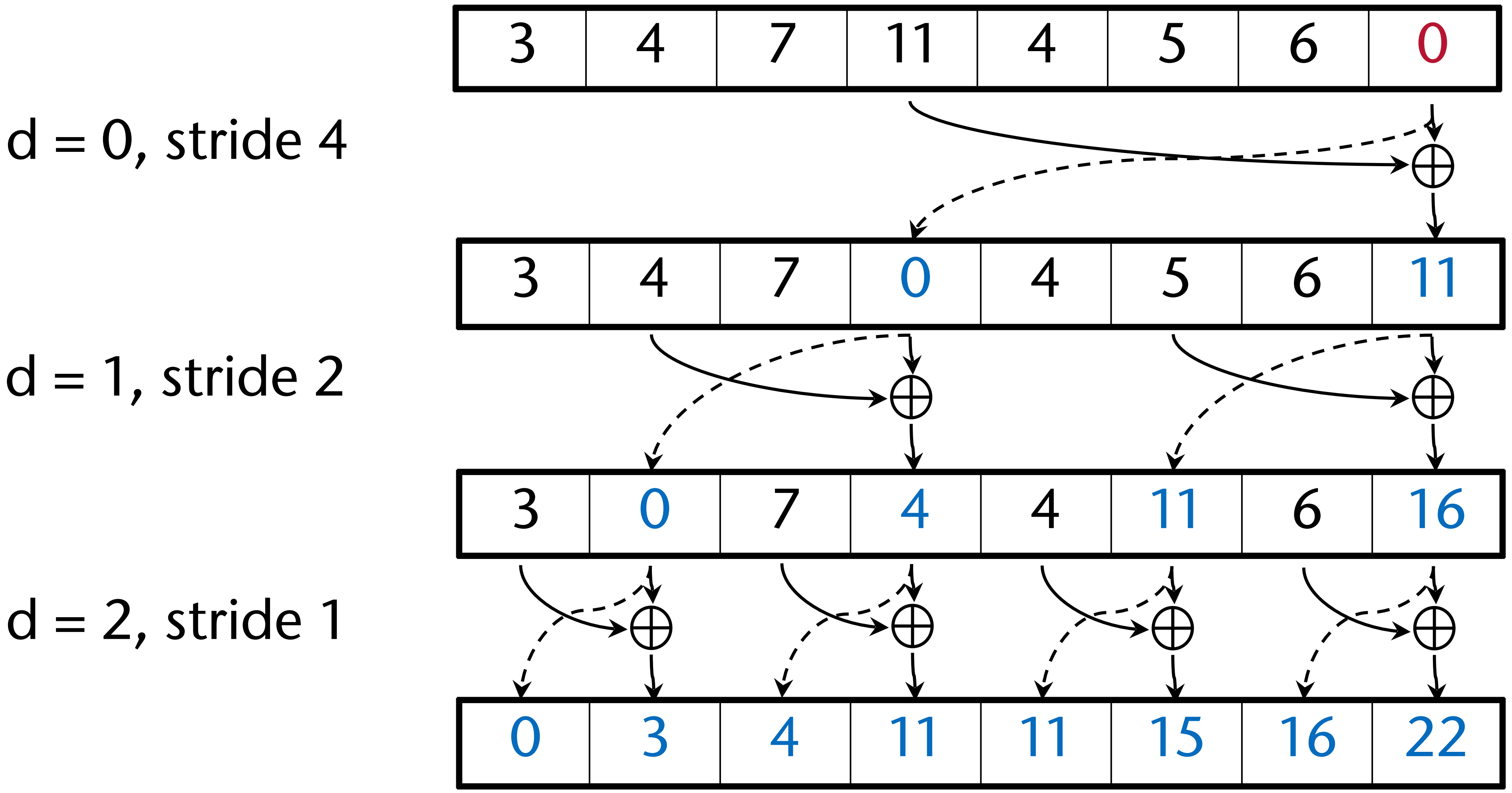
1. Up-sweep:



- Note: no double-buffering needed! (barrier sync is still needed, of course)

2. Down-sweep:

- First: zero last element (might seem strange at first thought)

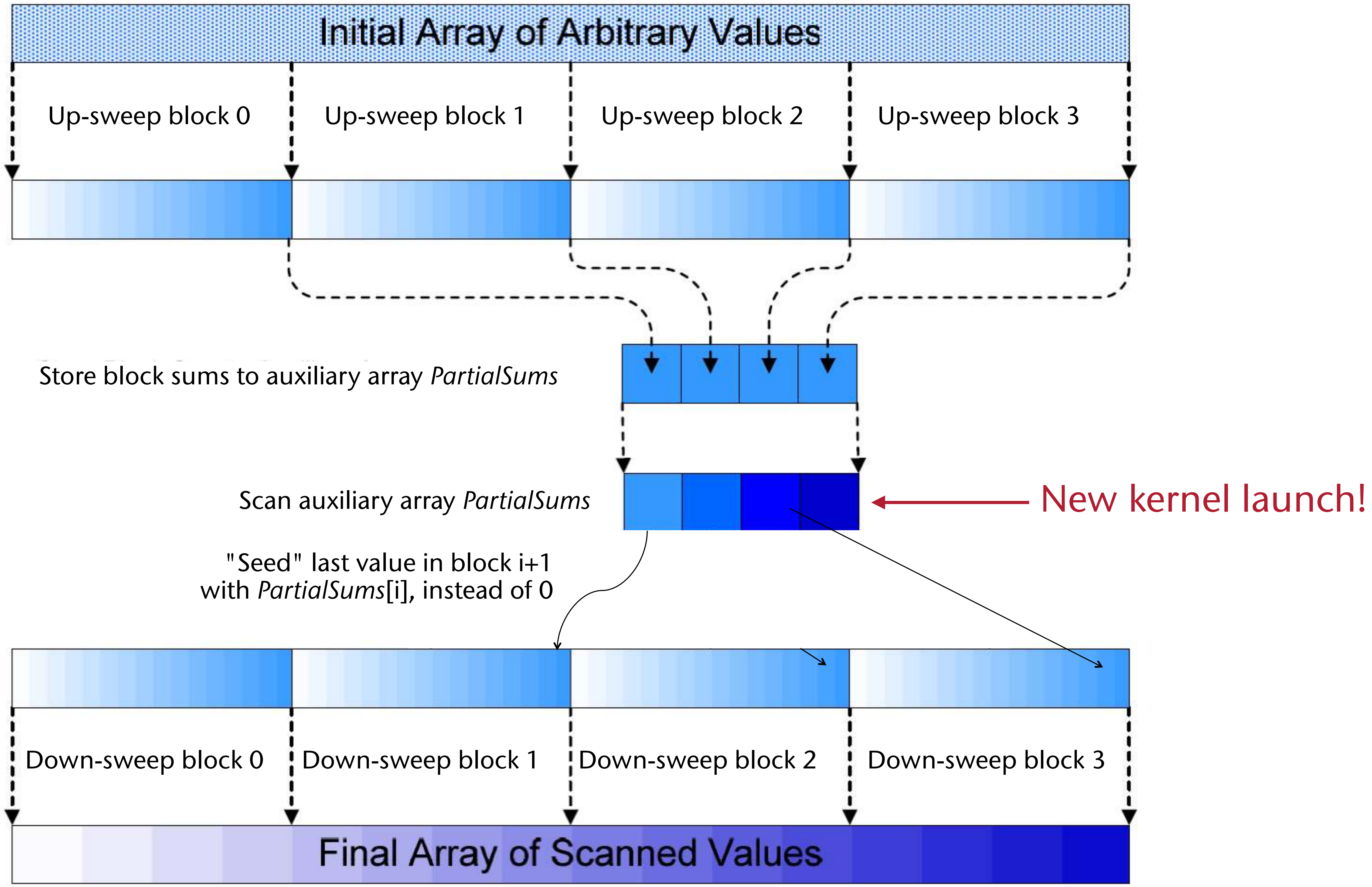


- Dashed line means "copy over" (overwriting previous content)

- Depth complexity:
  - Performs $2 \cdot \log(n)$ iterations
  - $D(n) \in O(\log n)$
- Work efficiency:
  - Number of adds: $n/2 + n/4 + .. + 1 + 1 + \ldots + n/4 + n/2$
  - Work complexity $W(n) = 2 \cdot n = O(n)$
  - The Blelloch algorithm is *work efficient*
- This *up-sweep followed by down-sweep* is a very common pattern in massively parallel algorithms!
- Limitations so far:
  - Only one block of threads (what if the array is larger?)
  - Only arrays with power-of-2 size

Working on Arbitrary Length Input

- Challenge: **`syncthreads()`** works only for all threads within a block, but NOT across block borders!
- Partition array into *b* blocks
  - Choose fairly small block size = 2^k, so we can easily pad array to $b \cdot 2^k$
- Run up-sweep on each block
- Each block writes the *sum of its partition* (= last element after up-sweep) into a **`PartialSums`** array at **`blockIdx.x`**
- Run prefix sum on the **`PartialSums`** array
- Perform down-sweep on each block
- Add **`PartialSums[blockIdx.x]`** to *each* element in "next" array section **`blockIdx.x+1`**


Initial Array of Arbitrary Values
Up-sweep block 0
Up-sweep block 1
Up-sweep block 2
Up-sweep block 3
Store block sums to auxiliary array PartialSums
Scan auxiliary array PartialSums
New kernel launch!
"Seed" last value in block i+1
with PartialSums[i], instead of 0
Down-sweep block 0
Down-sweep block 1
Down-sweep block 2
Down-sweep block 3
Final Array of Scanned Values

Further Simple & Effective Optimization

- Each thread **`i`** loads 4 floats from global memory → **`float4 x`**
- Store $\sum_{j=0\ldots3} x[i][j]$ in shared memory ⟶ **`a[i]`**
- Compute the exclusive prefix-sum on **`a`** ⟶ **`â`**
- Each thread **`i`** stores 4 values back in global memory:
  - **`A[4*i]   = â[i] + x[0]`**
  - **`A[4*i+1] = â[i] + x[0] + x[1]`**
  - **`A[4*i+2] = â[i] + x[0] + x[1] + x[2]`**
  - **`A[4*i+3] = â[i] + x[0] + x[1] + x[2] + x[3]`**
- Experience shows: 2x faster
- But *why* does this improve performance? → Brent's theorem

Brent's Theorem

- Frequent assumption when formulating parallel algorithms: we have arbitrarily many processors
  - E.g., $O(n)$ many processors for input of size n
  - Kernel launch even reflects that:
    - Often, we run as many threads as there are input elements
    - I.e., CUDA/GPU provide us with this (nice) abstraction
- Real hardware: only has fixed number p of processors
  - E.g., on current GPUs: $p \approx$ 200–2000 (depending on viewpoint and architecture)
- Question: how fast can an implementation of a parallel algorithm really be?

- Assumptions for Brent's theorem: PRAM model
  - No explicit synchronization needed
  - Memory access = free (no cost)

- Brent's Theorem:
  Given a massively parallel algorithm *A*; let $D(n)$ = its depth (i.e., parallel time) complexity, and $W(n)$ = its work complexity.
  Then, *A* can be run on a *p*-processor PRAM in time at most

$$T(n,p) \leq \left\lfloor \frac{W(n)}{p} \right\rfloor + D(n)$$

(Note the "≤")

- Alternative statement of Brent's theorem:

$$T_p(n) = \frac{T_1(n)}{p} + T_\infty(n)$$

where $T_p(n)$ = time complexity using p processors, T_1 = sequential complexity, T_∞ = parallel complexity with unlimited number of processors.

Proof

- For each iteration step i, $1 \leq i \leq D(n)$, let $W_i(n)$ = number of operations in that step
- In each iteration, distribute those $W_i(n)$ operations on p processors:
  - Execute $\left\lceil \frac{W_i(n)}{p} \right\rceil$ operations on each of the p processors in parallel
  - Takes $\left\lceil \frac{W_i(n)}{p} \right\rceil$ time steps on the PRAM
- Overall :

$$T(n,p) = \sum_{i=1}^{D(n)} \left\lceil \frac{W_i(n)}{p} \right\rceil \leq \sum_{i=1}^{D(n)} \left(\left\lfloor \frac{W_i(n)}{p} \right\rfloor + 1 \right) \leq \left\lfloor \frac{W(n)}{p} \right\rfloor + D(n)$$

Application of Brent's Theorem to our Optimization of Prefix-Sum

- Assume that the optimized version loads f floats into local registers
- Work complexity:
  - Without optimization: $W_1(n) = 2n$
  - With optimization: $W_2(n) = 2\frac{n}{f} + \frac{n}{f} \cdot f = n\left(1 + \frac{2}{f}\right)$
- Depth complexity:
  - Without optimization: $D_1(n) = 2\log(n)$
  - With optimization: $D_2(n) = 2\log(\frac{n}{f}) + 2f = 2\log n - 2\log f + 2f$
- If $f = 2$, then $W_2 = W_1$ and $D_2 = D_1$, i.e., we gain nothing
- If $f > 2$, speedup of version 2 (optimized) over version 1 (original):

$$\text{Speedup}(n) = \frac{T_1(n)}{T_2(n)} = \frac{\frac{W_1(n)}{p} + D_1(n)}{\frac{W_2(n)}{p} + D_2(n)} \approx \frac{2\frac{n}{p}}{\frac{n}{p}\left(1 + \frac{2}{f}\right)} = \frac{2f}{f+2}$$

Other Consequences of Brent's Theorem

- Obviously, $\text{Speedup}(n) \leq p$
- In the sequential world, time = work: $T_S(n) = W_S(n)$
- In the parallel world: $T_P(n) = \frac{W_P(n)}{p} + D(n)$
- Our speedup is $\text{Speedup}(n) = \frac{T_S(n)}{T_P(n)} = \frac{W_S(n)}{\frac{W_P(n)}{p} + D(n)}$
- Assume, $W_P(n) \in \Omega(\, W_S(n)\,)$

  i.e., our parallel algorithm would do asymptotically more work
- Then, $\text{Speedup}(n) = \frac{W_S(n)}{\Omega(\, W_S(n)\,) + D(n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$

  because, on real hardware, p is bounded
- This is the reason why we want work-efficient parallel algorithms!

- Now, look at work-efficient parallel algorithms, i.e. $W_P(n) \in \Theta(W_S(n))$
- Then,

$$\text{Speedup}(n) = \frac{W(n)}{\frac{W(n)}{p} + D(n)}$$

- In this situation, we will achieve the optimal speedup of O(*p*), so long as

$$p \in O\left(\frac{W(n)}{D(n)}\right)$$

- Consequence: given two work-efficient parallel algorithms, the one with the smaller depth complexity is better, because we can run it on hardware with more processors (cores) and still obtain a speedup of p over the sequential algorithm (in theory).
We say this algorithm scales better.

Limitations of Brent's Theorem

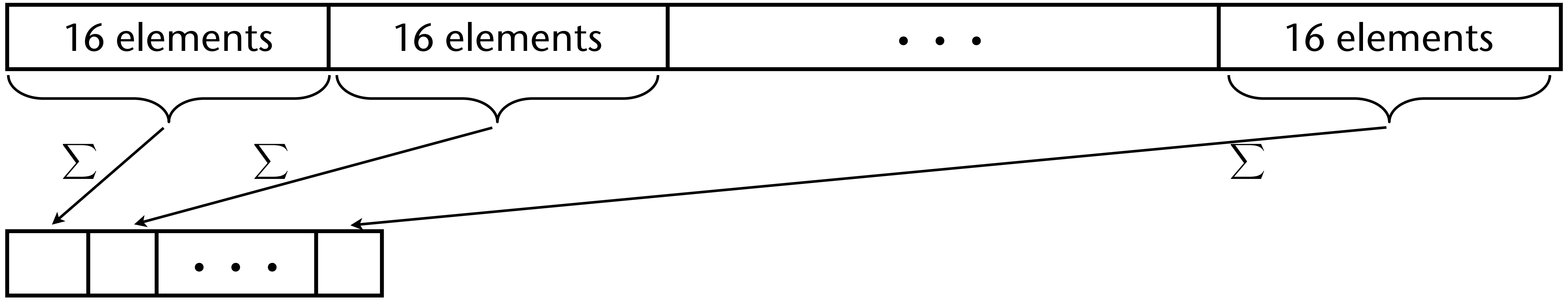
- Brent's theorem is based on the PRAM model
- That model makes a number of unrealistic assumptions:
  - Memory access has zero latency
  - Memory bandwidth is infinite
  - No synchronization among processors (threads) is necessary
  - Arithmetic operations cost unit time
- With current hardware, rather the opposite is realistic

Using Tensor Cores for Scan/Prefix Sum and Reduction

- Reduction ($\hat{a} = \sum_i a_i$) could be formulated as matrix multiplication:

$$\hat{a} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}}_{\text{P}} \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ a_2 & 0 & 0 & \dots & 0 \\ a_3 & \vdots & & & \\ a_n & 0 & 0 & \dots & 0 \end{pmatrix}$$

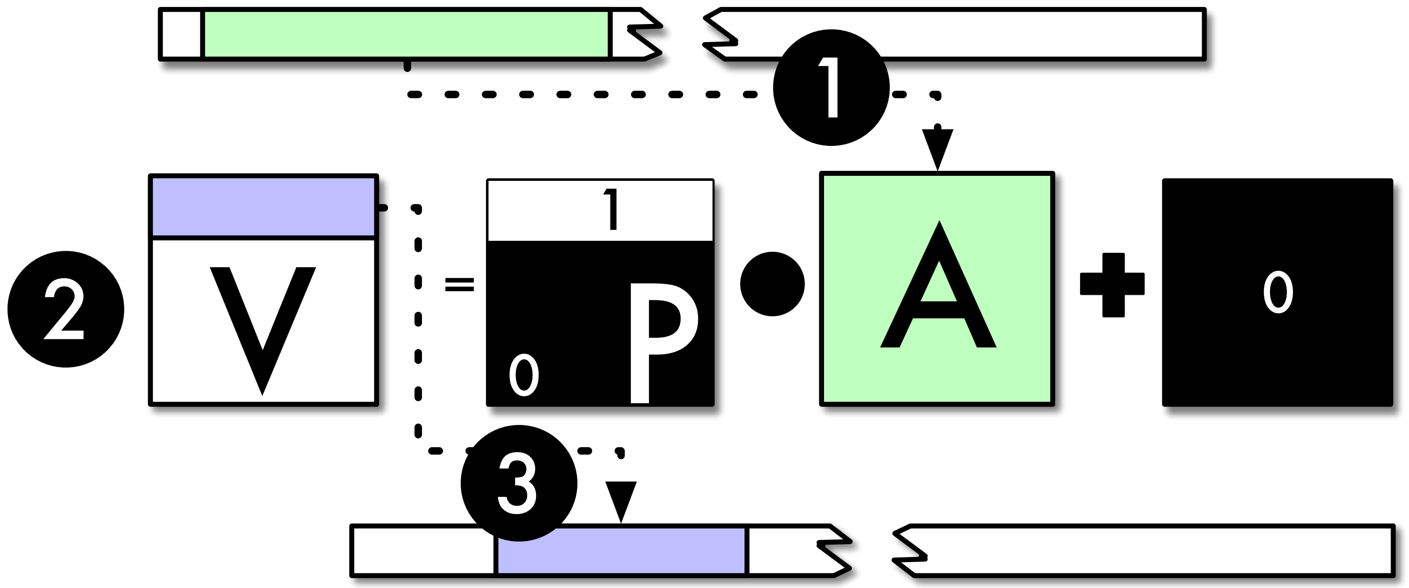
- Regular segmented reduction of length 16:



Algorithm for regular 16-segmented reduction

- Each warp loads parts of input array of size 256 = 16 segments of size 16, then performs a warp-level MMA (i.e., uses the tensor cores)

```
Reduction16( in array A, out array R ):

fragment_a ⟵ init matrix P
idx = global offset into A for each warp
fragment_b ⟵ load tile A[idx..idx+255] in column major
M = P·A + 0                          // = mma_sync() in CUDA
if lane index < 16:
  R[ idx/16 + lane-index ] = M[lane-index]
```



Extension to Scan Over 256 Elements

- Input: V = V[0], ..., V[255]
- Load V in to 16×16 matrix A in row-major order: $$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,16} \\ a_{2,1} & a_{2,2} & \dots & a_{2,16} \\ \vdots & & & \vdots \\ a_{16,1} & a_{16,2} & \dots & a_{16,16} \end{pmatrix}$$
- Define upper right 1-matrix: $$U = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$
- Multiplication yields row-wise inclusive scan, i.e., regular segmented inclusive prefix sum:

$$A \cdot U = \begin{pmatrix} a_{1,1} & \dots & \sum_{j=1}^{16} a_{1,j} \\ a_{2,1} & \dots & \sum_{j=1}^{16} a_{2,j} \\ \vdots & & \vdots \\ a_{16,1} & \dots & \sum_{j=1}^{16} a_{16,j} \end{pmatrix}$$

- Multiplication of A with lower-left 1-matrix (0's on the diagonal here!) yields a column-wise, exclusive prefix sum:

$$L = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix} \qquad L \cdot A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_{1,1} & a_{1,2} & \dots & a_{1,16} \\ a_{1,1} + a_{2,1} & a_{1,2} + a_{2,2} & \ddots & a_{1,16} + a_{2,16} \\ \sum_{j=1}^{15} a_{1,j} & \sum_{j=1}^{15} a_{2,j} & \dots & \sum_{j=1}^{15} a_{16,j} \end{pmatrix}$$

- Multiplication at right-hand side with an all-1-matrix yields row-wise reduction; all elements in the same row in the output matrix will be equal

$$J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

- Multiplication of $L \cdot A$ with J yields reduction of all elements in A *before* that row:

$$L \cdot A \cdot J = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \sum_{i=j}^{16} a_{1,j} & \sum_{i=j}^{16} a_{1,j} & \dots & \sum_{i=j}^{16} a_{1,j} \\ \vdots & & \ddots & \vdots \\ \sum_{j=1}^{16} a_{16,j} & \sum_{j=1}^{16} a_{16,j} & \dots & \sum_{j=1}^{16} a_{16,j} \end{pmatrix}$$

- Add the segmented scan $A \cdot U$, resulting in the inclusive prefix sum over 256 elements:

$$L \cdot A \cdot J + A \cdot U = \begin{pmatrix} a_{1,1} & a_{1,1} + a_{1,2} & \dots & \sum_{j=1}^{16} a_{1,j} \\ \sum_{j=1}^{16} a_{1,j} + a_{2,1} & \sum_{j=1}^{16} a_{1,j} + a_{2,1} + a_{2,2} & \dots & \sum_{i=1}^{2} \sum_{j=1}^{16} a_{i,j} \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^{15} \sum_{j=1}^{16} a_{i,j} + a_{16,1} & \sum_{i=1}^{15} \sum_{j=1}^{16} a_{i,j} + a_{16,1} + a_{16,2} & \dots & \sum_{i=1}^{16} \sum_{j=1}^{16} a_{i,j} \end{pmatrix}$$

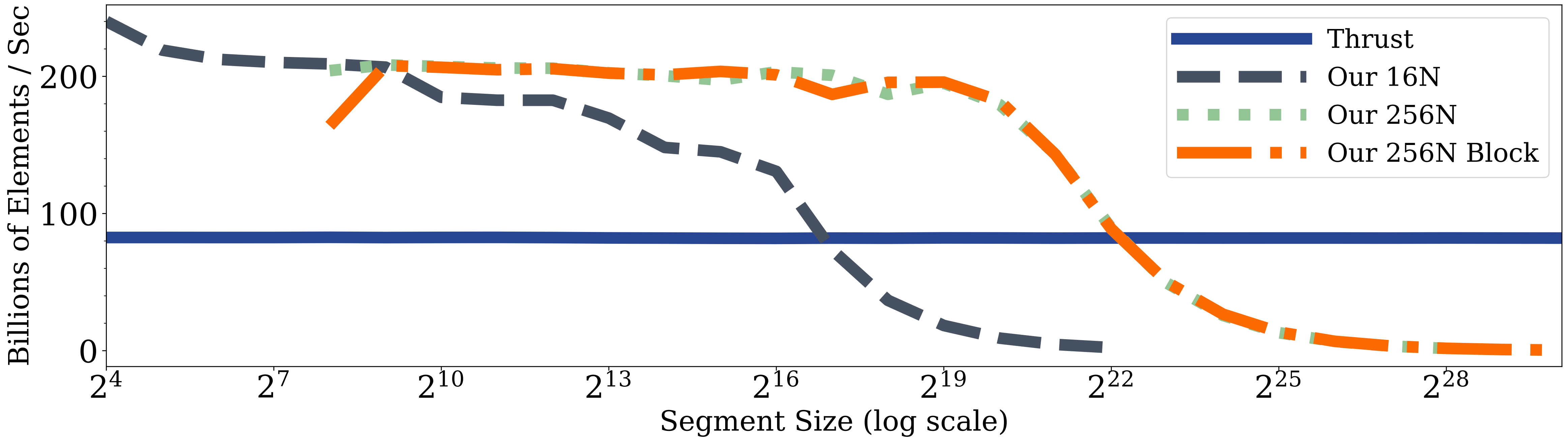
Extension to Scan Over Whole Block and Grid

- Per block: precondition is $N = b \cdot 256$, $b \leq 256$ (for sake of simplicity)

```
PrefixSumN( in array A, out array S ):

perform warp-level prefix-sum's over segments of A, 256 elements each
gather last element of each segment in array R
sync all threads within block
warp 0 performs exclusive prefix sum over R
sync all threads within block
all threads add R[warpIdx-1] to "their" element and output it to S
```

- Per grid:
  - Launch 3 kernels for 3 phases, similar to above procedure
  - First, block-wise (i.e., segmented) scan, gather last values of each segment (= reduced blocks) in intermediate array; second, prefix-sum over those values; third, distribute and accumulate values from intermediate scan to blocks

Performance for Segmented Prefix Sum



$N = 2^{31}$ elements

89% - 97% of theoretical peak throughput

Digression: Radix-Sort

- Modeled after sorting machines of post routing centers (but with a twist!)
- Disadvantages:
  - Not generic like Quicksort, which require only a *compare* operator on pairs of elements
  - Works only on elements with a known, pre-defined, fixed-length numeric representation (e.g., 32 bits)
  - Different representations require different versions of radix sort
- Advantage: very efficient!

- Observation: integers can be represented with any base r
- Naive (intuitive) idea:
  - Sort all elements according to the most significant digit into bins (one bin per digit)
  - Sort bin 0 using radix sort recursively
  - Sort bin 1 recursively, etc. …
- This is called MSD radix sort (MSD = *most significant digit*)
- For the algorithm on the next slide:
  - Choose radix r and fix it
  - Define $z(t,a)$ = t-th digit of number a when represented over base r, where t=0 denotes the least significant digit (usually the right-most digit)

The Algorithm (in Python)

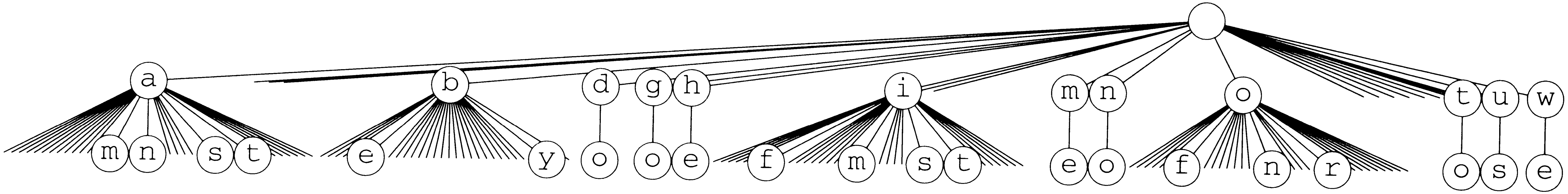
Optional

```
A = array of numbers
i = current digit used for sorting ( 0 <= i <= d-1 )
d = total number of digits (same for all keys)
def msd_radix_sort( A, i, d ):

    # init array of r empty lists = [ [], [], [], … ]
    bin = r * [[]]

    # distribute all A's in bins according to z(i,.)
    for j in range(0, len(A) ):
        bin[ z(i, A[j]) ].append( A[j] )

    # sort bins
    if i >= 0:
        for j in range(0, r):
            msd_radix_sort( bin[j], i-1, d )

    # gather bins
    A = []
    for j in range(0, r):
        A.extend( bin[j] )
        bin[j] = []
```

Example

- Keys = integers with 64 bits
- Size of input = 2^{24} (ca. 16m)
- We choose $r = 2^8 = 256$ as base
  - E.g. "digits" = characters in fixed-length strings
- On the first recursion level, the algo checks the left-most byte of the keys and distributes each key into one of 256 bins
- Average (expected) size of the bins (assuming uniform distribution of the keys) = $2^{24} / 2^8 = 2^{16} = 65536$

- Recursion tree:



- Problem: in each recursion, we need to save *r*-1 many bins (the remaining bin is passed down to the recursively called function)
  - Lots of house keeping necessary
  - Solutions: either use marker arrays like with Counting Sort
  - Or, use arrays of lists (lots of allocations / deallocations)

Solution: LSD Radix-Sort (aka. Backward Radix-Sort)

- First, sort according to least-significant digit, then according to least but second digit, etc.; do all of this *in place*, no auxiliary arrays needed!
- Let d = number of digits, digit 0 = least-significant one
- The algorithm:

```
lsd_radix_sort( A ):
  for i = 0, ..., d-1:
    do a stable sort on A with the
      i-th digit of the elements as the key
```

  - Use, e.g., Counting Sort inside the loop (check your Data Structures & Algorithms course)

Example

- Sort 12 letters according to the post code (zip code)
- In the first iteration, consider only the last digit

Brief	1	nach	3 5 0 3 7	Marburg
Brief	2	nach	7 1 6 7 2	Marbach
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	5	nach	8 8 6 6 2	Überlingen
Brief	6	nach	7 9 6 9 9	Zell
Brief	7	nach	8 0 6 3 8	München
Brief	8	nach	8 0 6 3 7	München
Brief	9	nach	5 5 1 2 8	Mainz
Brief	10	nach	5 5 4 6 9	Simmern
Brief	11	nach	8 2 3 4 0	Feldafing
Brief	12	nach	8 2 3 2 7	Tutzing

Letters before the first iteration

Brief	11	nach	8 2 3 4	0	Feldafing
Brief	2	nach	7 1 6 7	2	Marbach
Brief	4	nach	3 5 2 8	2	Rauschenberg
Brief	5	nach	8 8 6 6	2	Überlingen
Brief	1	nach	3 5 0 3	7	Marburg
Brief	8	nach	8 0 6 3	7	München
Brief	12	nach	8 2 3 2	7	Tutzing
Brief	3	nach	3 5 2 8	8	Wohratal
Brief	7	nach	8 0 6 3	8	München
Brief	9	nach	5 5 1 2	8	Mainz
Brief	6	nach	7 9 6 9	9	Zell
Brief	10	nach	5 5 4 6	9	Simmern

Letters after the first iteration

- Notice: letters with the same digit did not change their position relative to each other!

- Sort by last but second digit

Brief	11	nach	8 2 3 4 0	Feldafing
Brief	2	nach	7 1 6 7 2	Marbach
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	5	nach	8 8 6 6 2	Überlingen
Brief	1	nach	3 5 0 3 7	Marburg
Brief	8	nach	8 0 6 3 7	München
Brief	12	nach	8 2 3 2 7	Tutzing
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	7	nach	8 0 6 3 8	München
Brief	9	nach	5 5 1 2 8	Mainz
Brief	6	nach	7 9 6 9 9	Zell
Brief	10	nach	5 5 4 6 9	Simmern

Letters before the second iteration

Brief	12	nach	8 2 3 2 7	Tutzing
Brief	9	nach	5 5 1 2 8	Mainz
Brief	1	nach	3 5 0 3 7	Marburg
Brief	8	nach	8 0 6 3 7	München
Brief	7	nach	8 0 6 3 8	München
Brief	11	nach	8 2 3 4 0	Feldafing
Brief	5	nach	8 8 6 6 2	Überlingen
Brief	10	nach	5 5 4 6 9	Simmern
Brief	2	nach	7 1 6 7 2	Marbach
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	6	nach	7 9 6 9 9	Zell

Letters after the second iteration

Brief	12	nach	8 2 3 2 7	Tutzing
Brief	9	nach	5 5 1 2 8	Mainz
Brief	1	nach	3 5 0 3 7	Marburg
Brief	8	nach	8 0 6 3 7	München
Brief	7	nach	8 0 6 3 8	München
Brief	11	nach	8 2 3 4 0	Feldafing
Brief	5	nach	8 8 6 6 2	Überlingen
Brief	10	nach	5 5 4 6 9	Simmern
Brief	2	nach	7 1 6 7 2	Marbach
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	6	nach	7 9 6 9 9	Zell

Brief	1	nach	3 5 0 3 7	Marburg
Brief	9	nach	5 5 1 2 8	Mainz
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	12	nach	8 2 3 2 7	Tutzing
Brief	11	nach	8 2 3 4 0	Feldafing
Brief	10	nach	5 5 4 6 9	Simmern
Brief	8	nach	8 0 6 3 7	München
Brief	7	nach	8 0 6 3 8	München
Brief	5	nach	8 8 6 6 2	Überlingen
Brief	2	nach	7 1 6 7 2	Marbach
Brief	6	nach	7 9 6 9 9	Zell

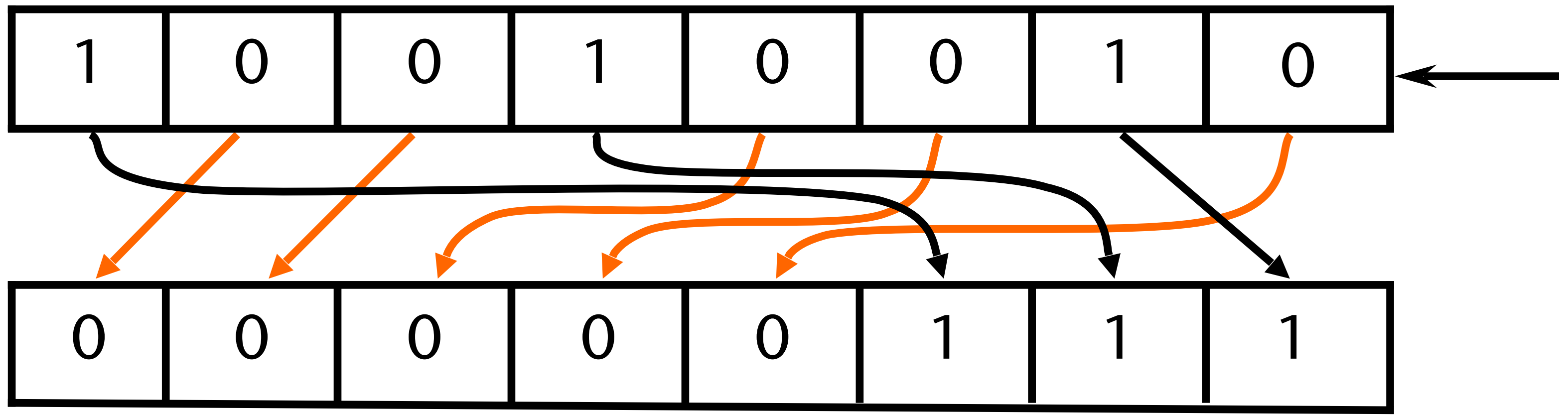
Brief	8	nach	8 0 6 3 7	München
Brief	7	nach	8 0 6 3 8	München
Brief	2	nach	7 1 6 7 2	Marbach
Brief	12	nach	8 2 3 2 7	Tutzing
Brief	11	nach	8 2 3 4 0	Feldafing
Brief	1	nach	3 5 0 3 7	Marburg
Brief	9	nach	5 5 1 2 8	Mainz
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	10	nach	5 5 4 6 9	Simmern
Brief	5	nach	8 8 6 6 2	Überlingen
Brief	6	nach	7 9 6 9 9	Zell

Brief	1	nach	3 5 0 3 7	Marburg
Brief	4	nach	3 5 2 8 2	Rauschenberg
Brief	3	nach	3 5 2 8 8	Wohratal
Brief	9	nach	5 5 1 2 8	Mainz
Brief	10	nach	5 5 4 6 9	Simmern
Brief	2	nach	7 1 6 7 2	Marbach
Brief	6	nach	7 9 6 9 9	Zell
Brief	8	nach	8 0 6 3 7	München
Brief	7	nach	8 0 6 3 8	München
Brief	12	nach	8 2 3 2 7	Tutzing
Brief	11	nach	8 2 3 4 0	Feldafing
Brief	5	nach	8 8 6 6 2	Überlingen

Letters after the fifth iteration

Parallel Radix Sort, Based on the Split Operation

- We can use base=2 (radix=2); nice consequence: we only need to maintain 2 bins, and we can re-use the input array to hold both bins
- The split operation: rearrange elements according to a flag



Flags, i.e., the *i*-th bits of the keys (Usually, there are payload data, too; omitted here)

- Note: split maintains order within each group! (i.e., it is *stable*)
- Use double buffering to prevent expensive synchronization among threads

- Radix sort (massively parallel):

```
radix_sort( array a, int len ):
  for i = 0...numbits-1:  // important: go from low to high bit!
    split(i, a)           // split a, based on bit i of the keys
```

where **split(i,a)** rearranges **a** by moving all keys that have bit **i** = 0 to the front, and all keys that have bit **i** = 1 to the back (bit 0 = LSB)

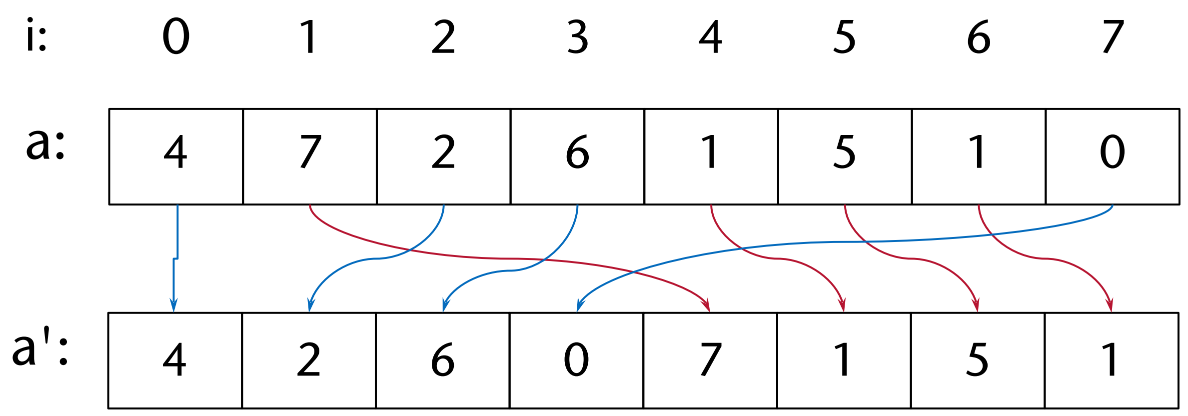
- Reminder: stability of split is essential!
- Note: main job of the split operation is to compute "which key goes where"
- Hint: the prefix-sum is probably up to the job :-)

Algorithm for the Massively-Parallel Split Operation

- Split's job:
  - Determine new index for each element
  - Then perform the permutation (stable!)
- Algorithm (by way of the example):
  - Consider lowest bit of the keys

1. Compute exclusive "0"-scan: f_i = # 0's in $(a_0, \ldots, a_{i-1})$
2. Set F = total number of 0's $= \begin{cases} f_{n-1}+1 & , a_{n-1}=0 \\ f_{n-1} & , a_{n-1}=1 \end{cases}$
3. Construct d = new positions of the a_i's
   - If a_i's bit = 0 → new position $d_i = f_i$
   - If a_i's bit = 1 → new position $d_i = F + (i - f_i)$, because $i - f_i$ = # 1's to the left of a_i

Example: split based on bit 0



i:	0	1	2	3	4	5	6	7
a:	100	111	010	110	001	101	001	000
f:	0	1	1	2	3	3	3	3
$d_{\text{for "0"s}}$:	0		1	2				3
$d_{\text{for "1"s}}$:		4+(1-1)			4+(4-3)	4+(5-3)	4+(6-3)	
d:	0	4	1	2	5	6	7	3

- A conceptual algorithm for the "0"-scan:
  - Extract the relevant bit (conceptually only)
  - Invert the bit
  - Compute regular prefix sum with "+" operation

a:	100	111	010	110	001	101	001	000
a':	1	0	1	1	0	0	0	1
f:	0	1	1	2	3	3	3	3

- In a real implementation, you would, of course, implement this as a native "0"-scan routine with a special "+" operation in the first iteration!
- Depth complexity:

$$O(\, b \cdot \log(n) \,), \quad \text{where } b = \text{\#bits per integer, and } n = \text{\# elements}$$

  - Amounts to $O(\, b^2 \,)$, or $O(\, \log^2(n) \,)$

Stream Compaction

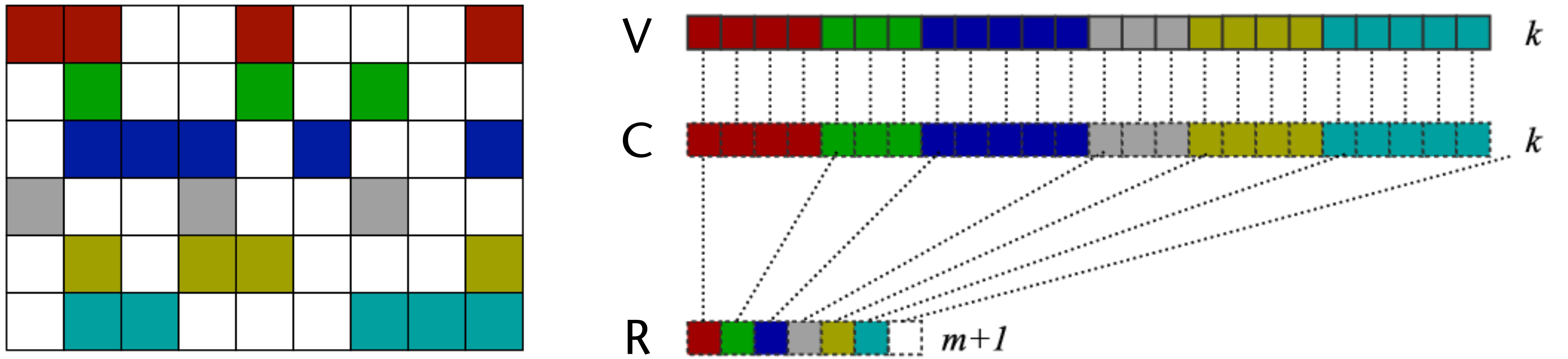
- Given: input stream A, and a *flag/predicate* for each a_i
- Goal: output stream A' that contains only a_i's, for which flag = true
- Example:
  - Given: array of upper and lower case letters
  - Goal: delete lower case letters and compact the others to the front of the array

a:	A	x	C	P	h	w	b	Z
a':	1	0	1	1	0	0	0	1
b:	A	C	P	Z				

- Solution:
  - Just like with the split operation, except we don't compute indices for the "to-be-deleted" elements
- Frequent task, sometimes A/flags are not given explicitly (e.g., collision detection)
- Sometimes also called list packing, or stream packing

Sparse Matrices

- "Unstructured" sparse matrices:
  - Most common storage format is Compressed Sparse Row (CSR)
  - Matrix M, size $m \times n$, k non-zero elements (a.k.a. "nnz")
  - Stored in three arrays V, C, R
    - Row i of matrix M is stored in $V_{R_i}, \ldots, V_{R_{i+1}-1}$
    - C contains column indices: element V_j in M's i-th row represents element M_{i,C_j}



Example

$$M = \begin{pmatrix} a_0 & 0 & 0 & a_1 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & a_4 & 0 & a_5 & a_6 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix}$$

$$V = (a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7)$$

$$C = (0,\ 3,\ 1,\ 2,\ 1,\ 3,\ 4,\ 4)$$

$$R = (0,\ 2,\ 3,\ 4,\ 7,\ 8)$$

- Implementation in C:

```
struct {
  int n_rows;              // number of rows
  int nnz;                 // = k = total number of non-zero elements
  int row_start[n_rows+1];
  int col_idx[nnz];
  double val[nnz];
}
```

where

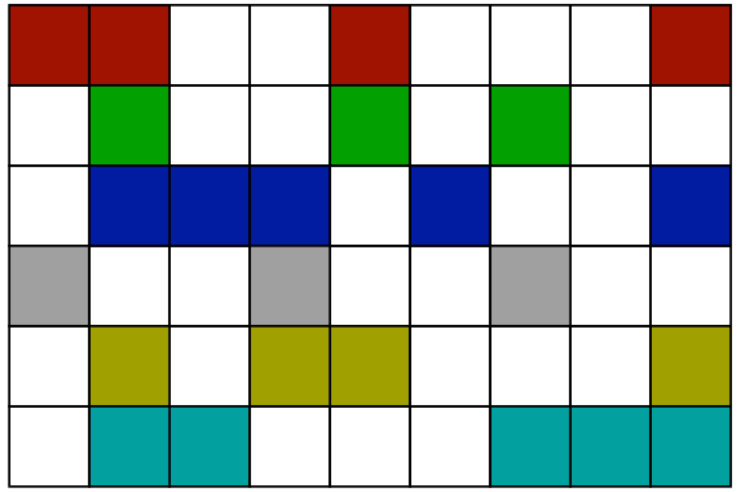
n_rows $= m$,

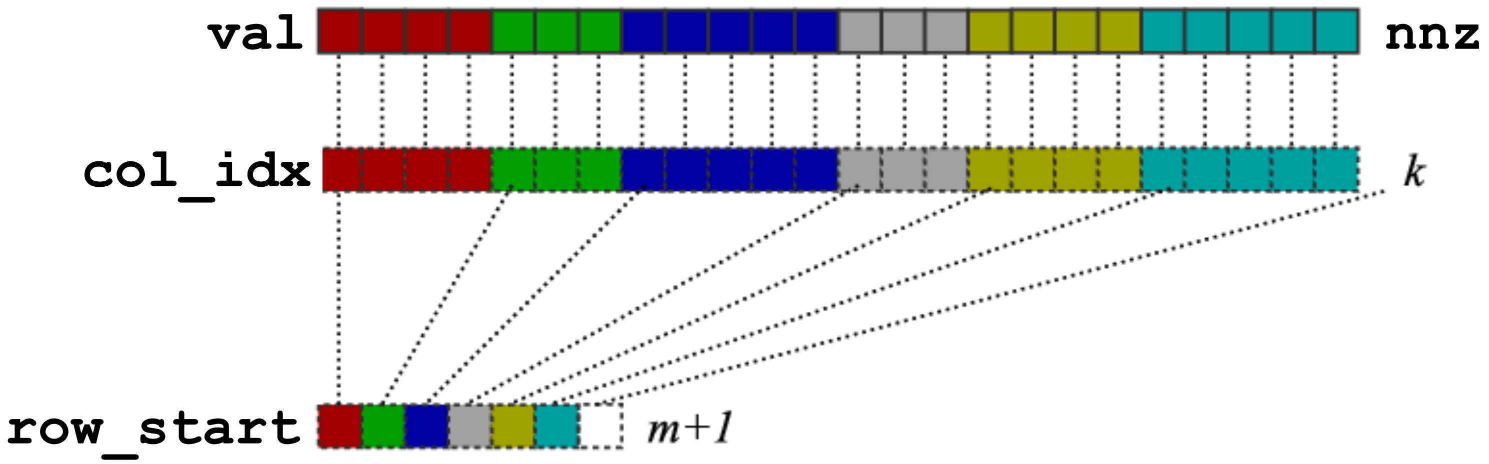
nnz $= k$,

val $= V$,

col_idx $= C$,

row_start $= R$





Sparse Matrix-Vector Multiplication (SPMV)

- Task: $y = Mx$, where M is given as CSR

$$V = (a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7)$$
$$C = (0,\ 3,\ 1,\ 2,\ 1,\ 3,\ 4,\ 4)$$
$$R = (0,\ 2,\ 3,\ 4,\ 7,\ 8,\ )$$

1. Multiply each element in V with its corresponding element in x:
$$V'_i = V_i \cdot x_{c_i}$$

$$V' = (a_0x_0,\ a_1x_3,\ a_2x_1,\ a_3x_2,\ a_4x_1,\ a_5x_3,\ a_6x_4,\ a_7x_4)$$

2. Compute flags array, signifying row starts:
$$F_i = 1 \Leftrightarrow i \in R$$

$$F = (1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1)$$

3. Inclusive segmented scan (one segment per row): V' $\longrightarrow$ V''

$$V'' = (a_0x_0,\ a_0x_0 + a_1x_3,\ a_2x_1,\ a_3x_2,\ a_4x_1,\ a_4x_1 + a_5x_3,\ a_4x_1 + a_5x_3 + a_6x_4,\ a_7x_4)$$

4. Retrieve elements for y: $y_i = V''_{R_{i+1}-1}$

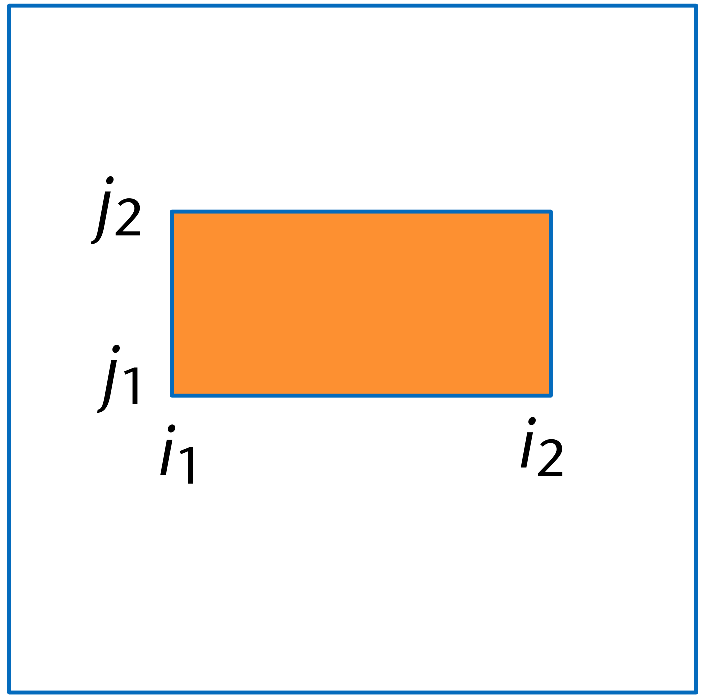
$$y_3 = V''_{R_4-1} = a_4x_1 + a_5x_3 + a_6x_4$$

Summed-Area Tables / Integral Images

- Given: 2D array T of size $w \times h$
- Wanted: a data structure that allows to compute

$$\sum_{k=i_1}^{i_2} \sum_{l=j_1}^{j_2} T(k, l)$$

for any i_1, i_2, j_1, j_2 in $O(1)$ time



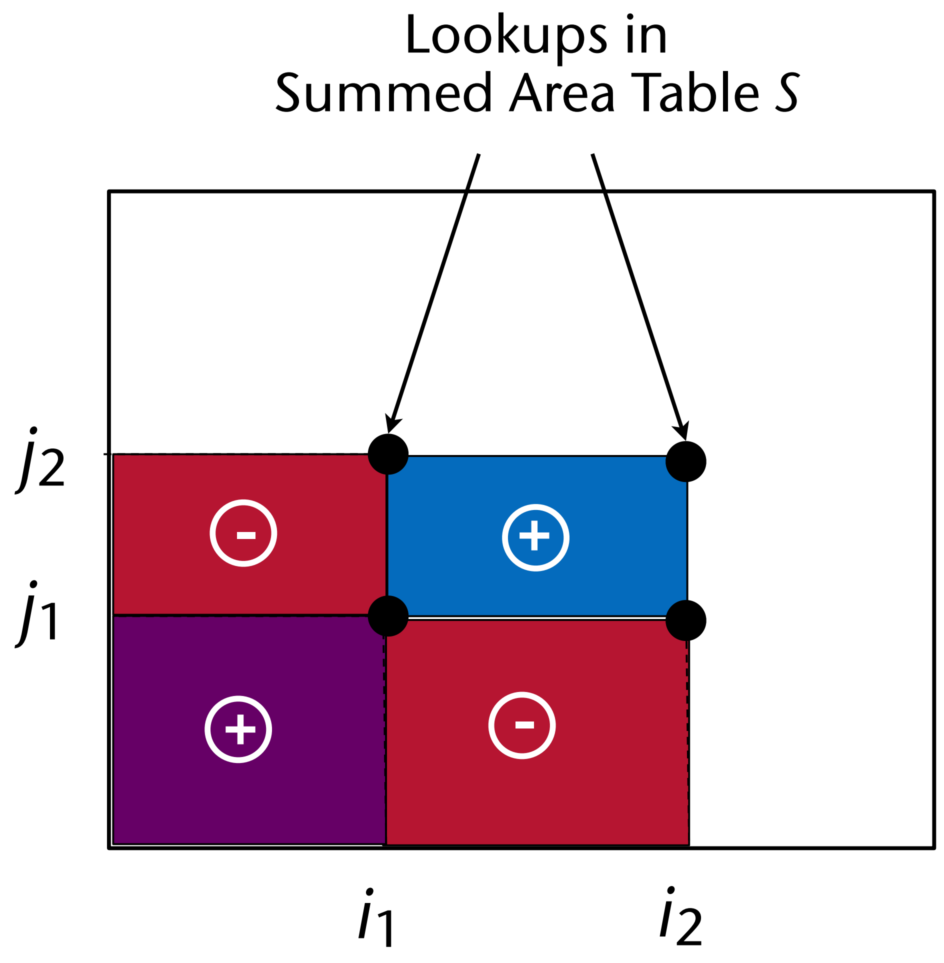
- The trick:

$$\sum_{k=i_1}^{i_2}\sum_{l=j_1}^{j_2} T(k,l) = \sum_{k=1}^{i_2}\sum_{l=1}^{j_2} T(k,l) - \sum_{k=1}^{i_1}\sum_{l=1}^{j_2} T(k,l) - \sum_{k=1}^{i_2}\sum_{l=1}^{j_1} T(k,l) + \sum_{k=1}^{i_1}\sum_{l=1}^{j_1} T(k,l)$$

- Define $S(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} T(k,l)$

- With that, we can rewrite the sum:

$$\sum_{k=i_1}^{i_2}\sum_{l=j_1}^{j_2} T(k,l) = S(i_2,j_2) - S(i_1,j_2) - S(i_2,j_1) + S(i_1,j_1)$$



- Definition:
  Given a 2D array of numbers, *T*, the summed area table *S* stores for each index (*i*,*j*) the sum of all elements in the rectangle (0,0) and (*i*,*j*) (inclusively):

$$S(i,j) = \sum_{k=1}^{i} \sum_{l=1}^{j} T(k,l)$$

- Like the prefix-sum, but for higher dimensions
  - Summed area tables can also be defined for higher dimensions
- In computer vision, it is often called integral image
- Example:

Input

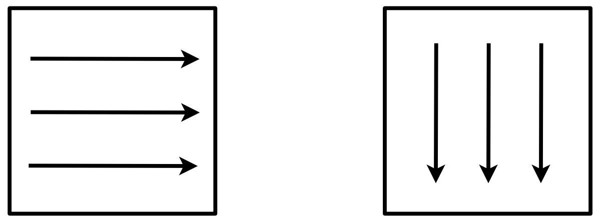
2	1	0	0
0	1	2	0
1	2	1	0
1	1	0	2

Summed Area Table

4	9	12	14
2	6	9	11
2	5	6	8
1	2	2	4

The Algorithm

- 2 phases (for 2D)
  1. Do h prefix-sums horizontally (one per kernel launch)
  2. Do w prefix-sums vertically (ditto)
     - In order to maintain *coalesced memory access*): horizontal scan, transpose img., horiz. scan
     - Or use texture memory (?)



- Depth complexity for d dimensions, $w = h$, and ignore transposit $d \cdot w^{d-1} \log w$
- Caveat: beware of precision loss in integer/floating-point arithmetic
  - Assumption: each T_{ij} needs b bits
  - Consequence: number of bits needed for $S_{wh} = \log w + \log h + b$
  - Example: 1024x1024 grey scale image, each pixel = 8 bits → ≥28 bits needed in S

Increasing the Precision

- The following techniques actually apply to prefix-sums, too!

1. "Signed offset" representation:
   - Set $T'(i,j) = T(i,j) - \bar{t}$
     where $\bar{t} = \text{average of } T = \frac{1}{wh} \sum_{1}^{w} \sum_{1}^{h} T(i,j)$
   - Effectively "removes the DC component from the signal"
   - Consequence:

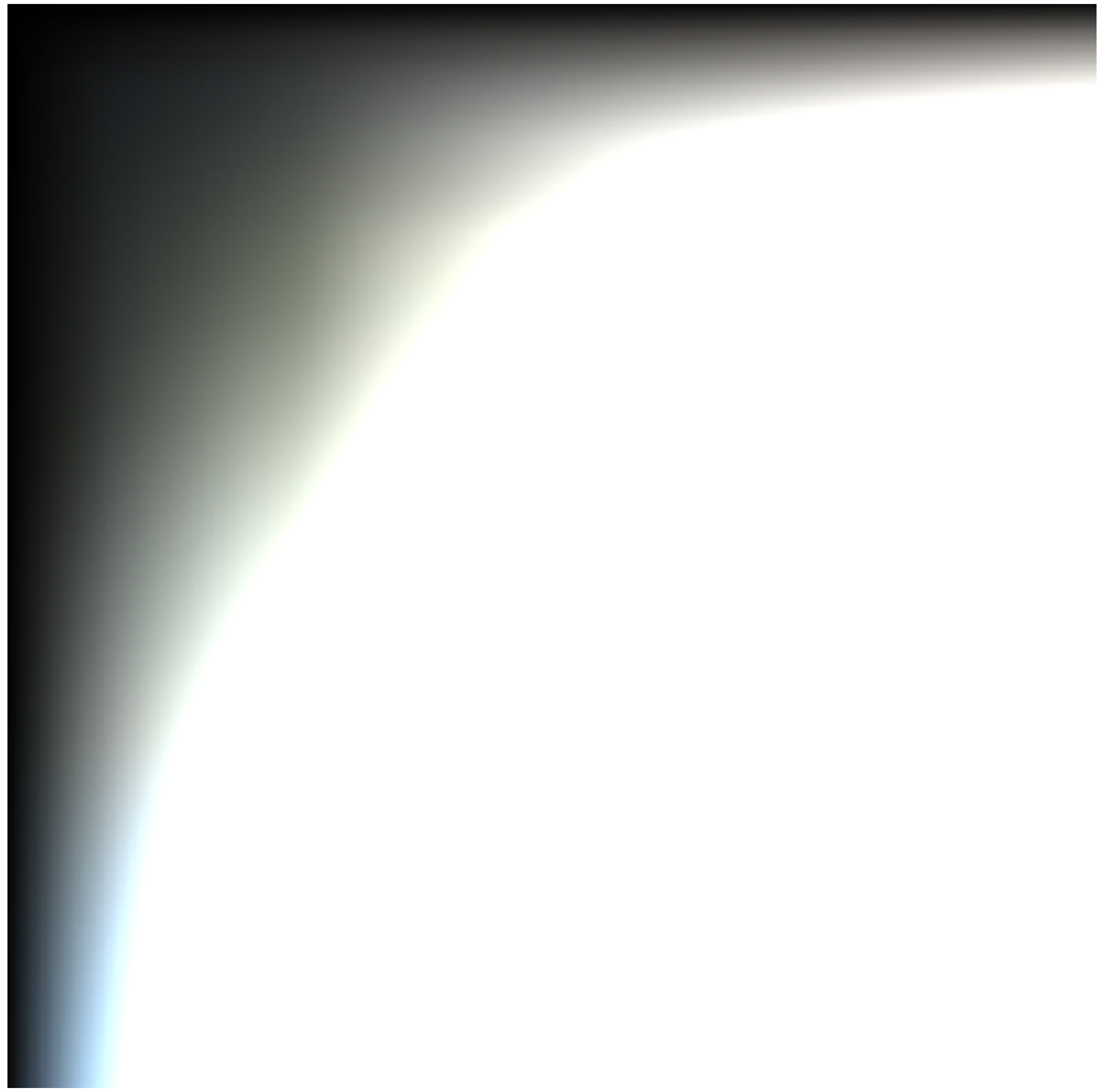
$$S'(i,j) = \sum_{k=1}^{i} \sum_{l=1}^{j} T'(k,l) = S(i,j) - i \cdot j \cdot \bar{t}$$

     i.e., the values of S' are now in the same order as the values of T (less bits have to be thrown away during the summation)
   - Note 1: we need to set aside 1 bit (sign bit)
   - Note 2: $S'(w,h) = 0$ (modulo rounding errors)

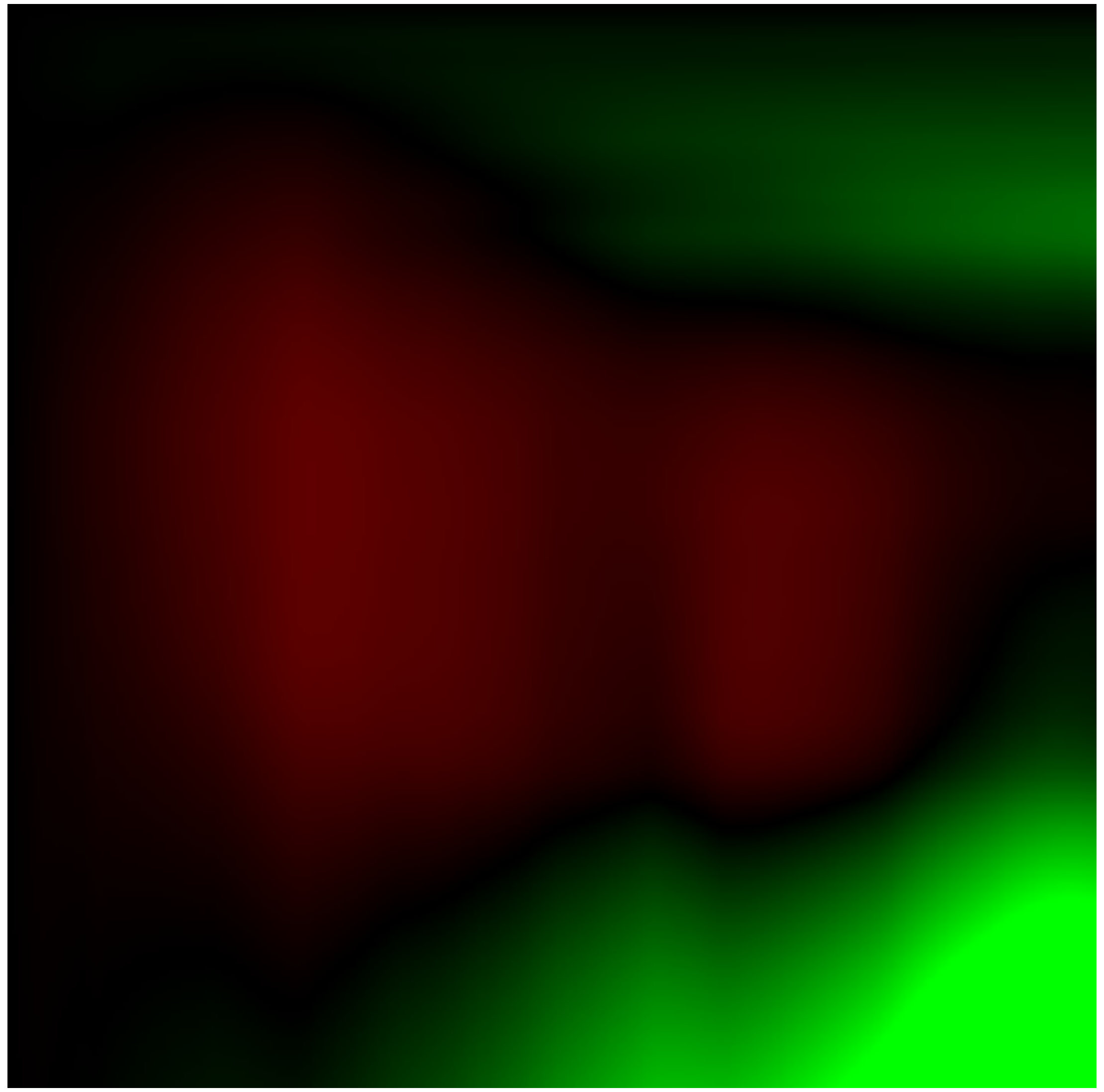
Example

Input image

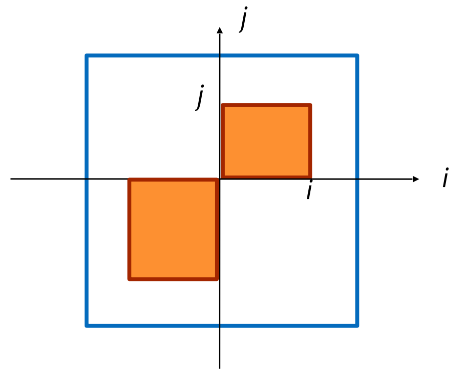
Original summed area table



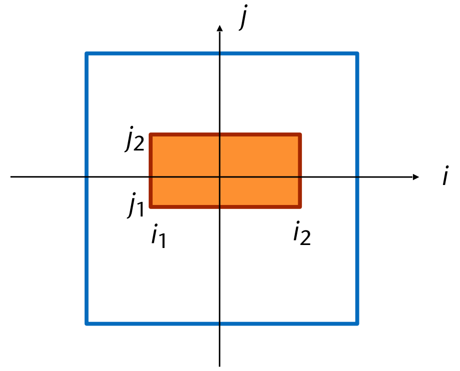
With improved precision
using "offset" representation



- Move the "origin" of the *i,j* "coordinate frame":
  - Compute 4 different *S*-tables, one for each quadrant
  - Result: each *S*-table comprises only ¼ of the pixels/values of *T*



- For computation of $\sum_{k=i_1}^{i_2} \sum_{l=j_1}^{j_2} T(k,l)$ do a simple case switch



Results

- Compute integral image
- From that, compute

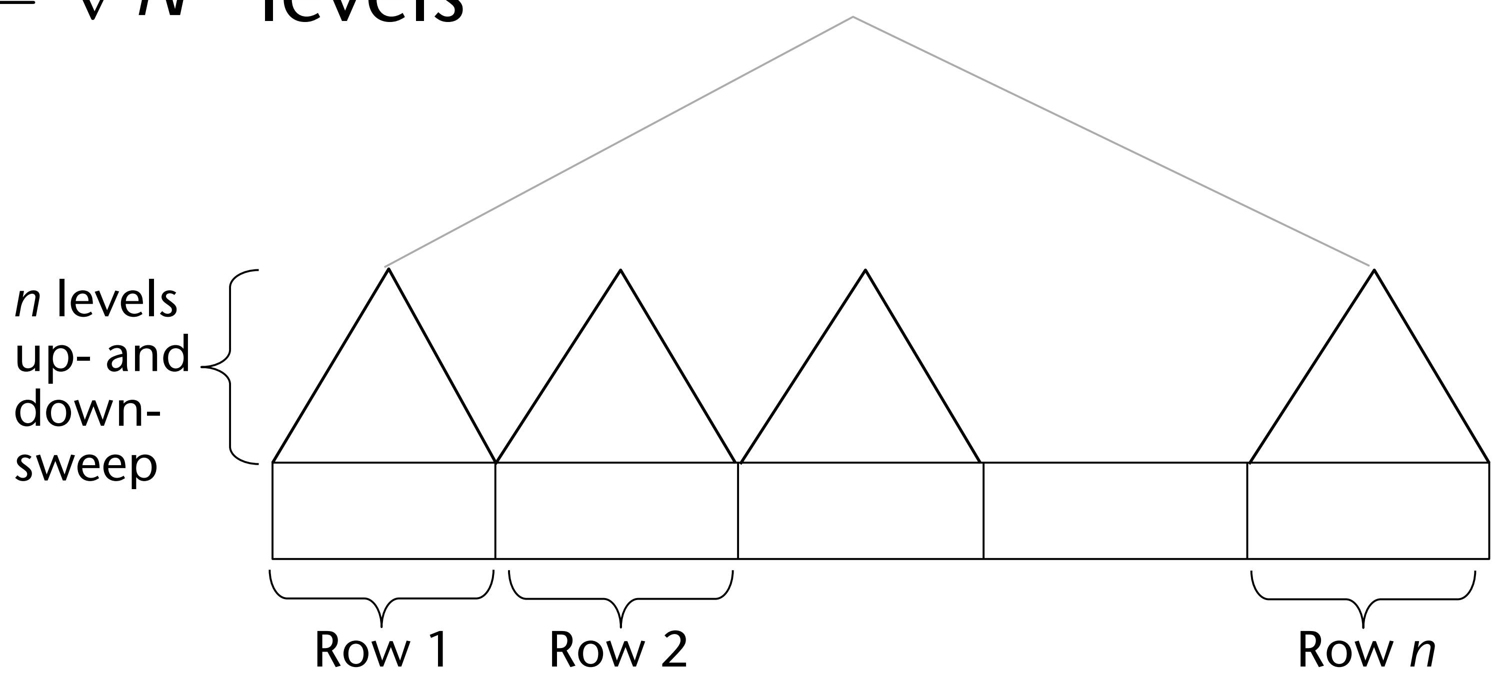
$$
\begin{aligned}
&S(i,j) \\
-&S(i-1,j) \\
-&S(i,j-1) \\
+&S(i-1,j-1)
\end{aligned}
$$

- Should yield the original image (theoretically)

With methods 1 & 2 | Simple method

Efficient Computation of the Integral Image

- Assumption: image = N pixels
- Naïve approach: do a 1D prefix-sum per row (no transposition step)
  - Depth complexity: $O\left(\sqrt{N} \log N\right)$
  - Work complexity: $O(\sqrt{N} \cdot \sqrt{N}) = O(N)$
- Better solution:
  - Pack all rows into one linear array of size N
  - Do a 1D prefix-sum, but stop after the first $n = \sqrt{N}$ levels
  - Depth complexity $= O\left(\log N\right)$
  - Work complexity = $O\left(N\right)$
- Is a special case of segmented prefix sum

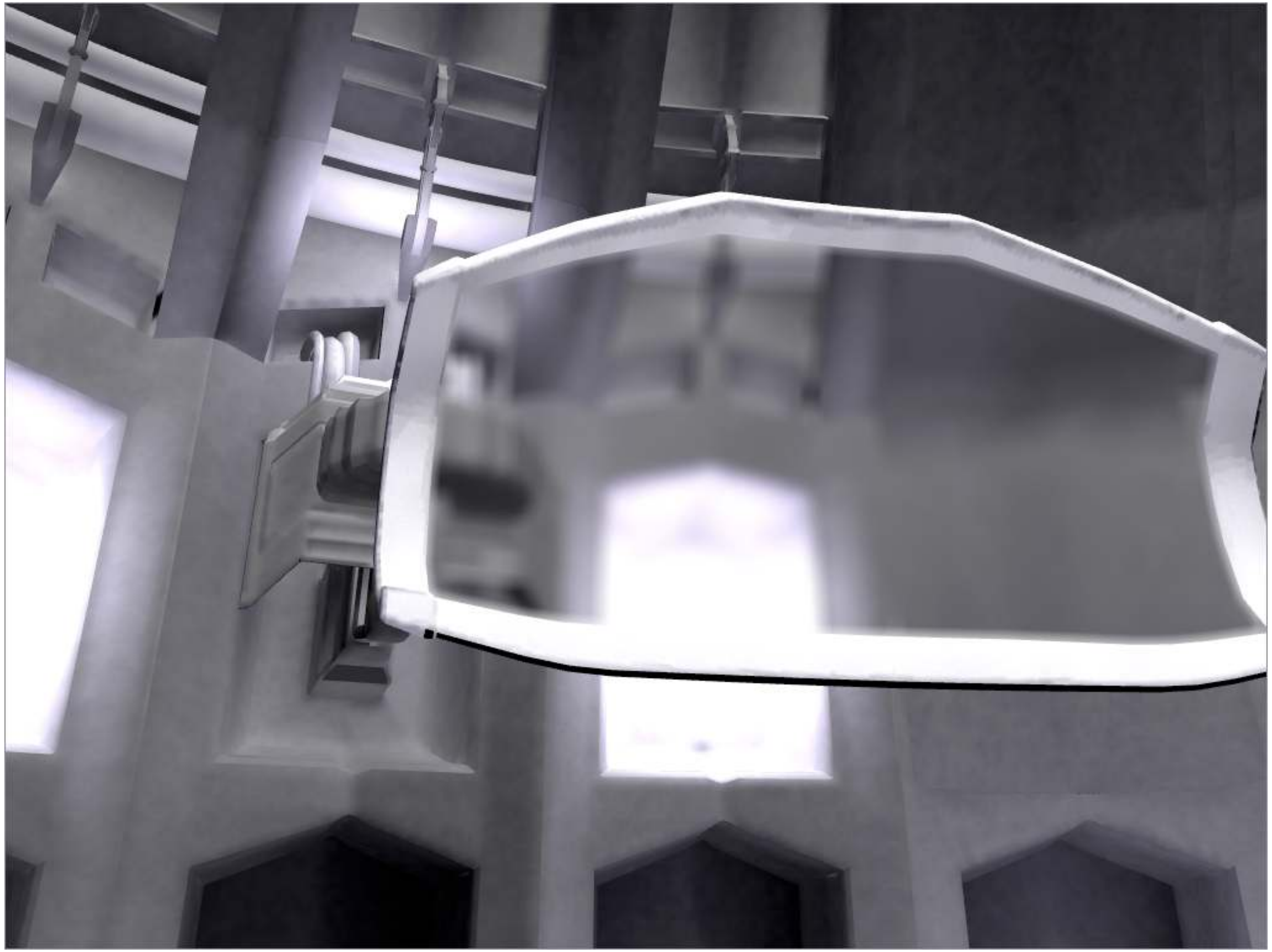


Applications of the Summed Area Table

- For filtering in general
- Simple example: box filter (blurring)
  - Slide box across image (convolution)
  - Compute average inside a box (= rectangle)
- Application: translucent objects, i.e., transparent & matte
  - E.g., "simulate" milky glass object in a game

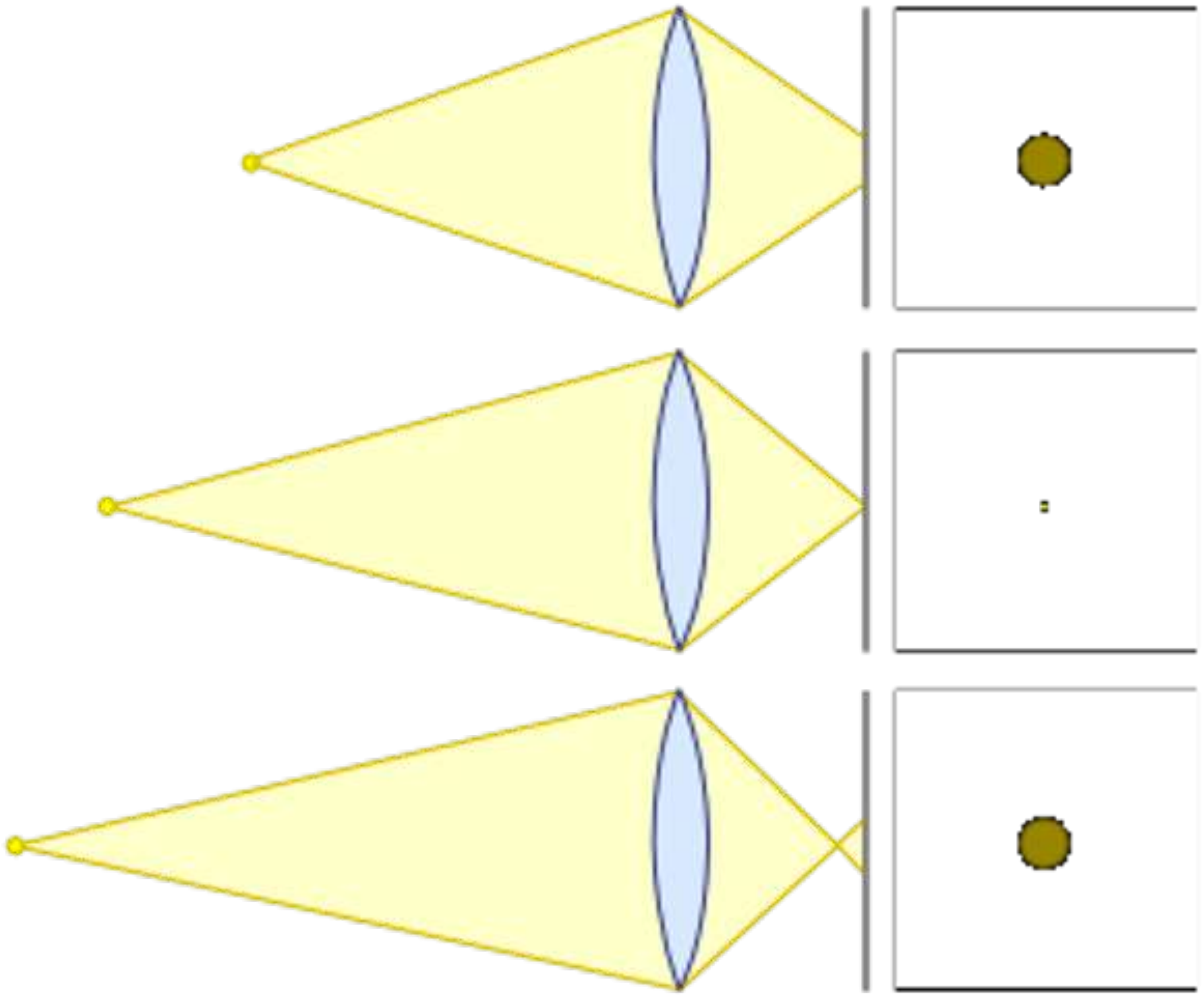
1. Render virtual scene without translucent objects
2. Compute summed area table from frame buffer
3. Render translucent object (using a fragment shader): replace pixel behind translucent object by average over original image within a (small) box

Result



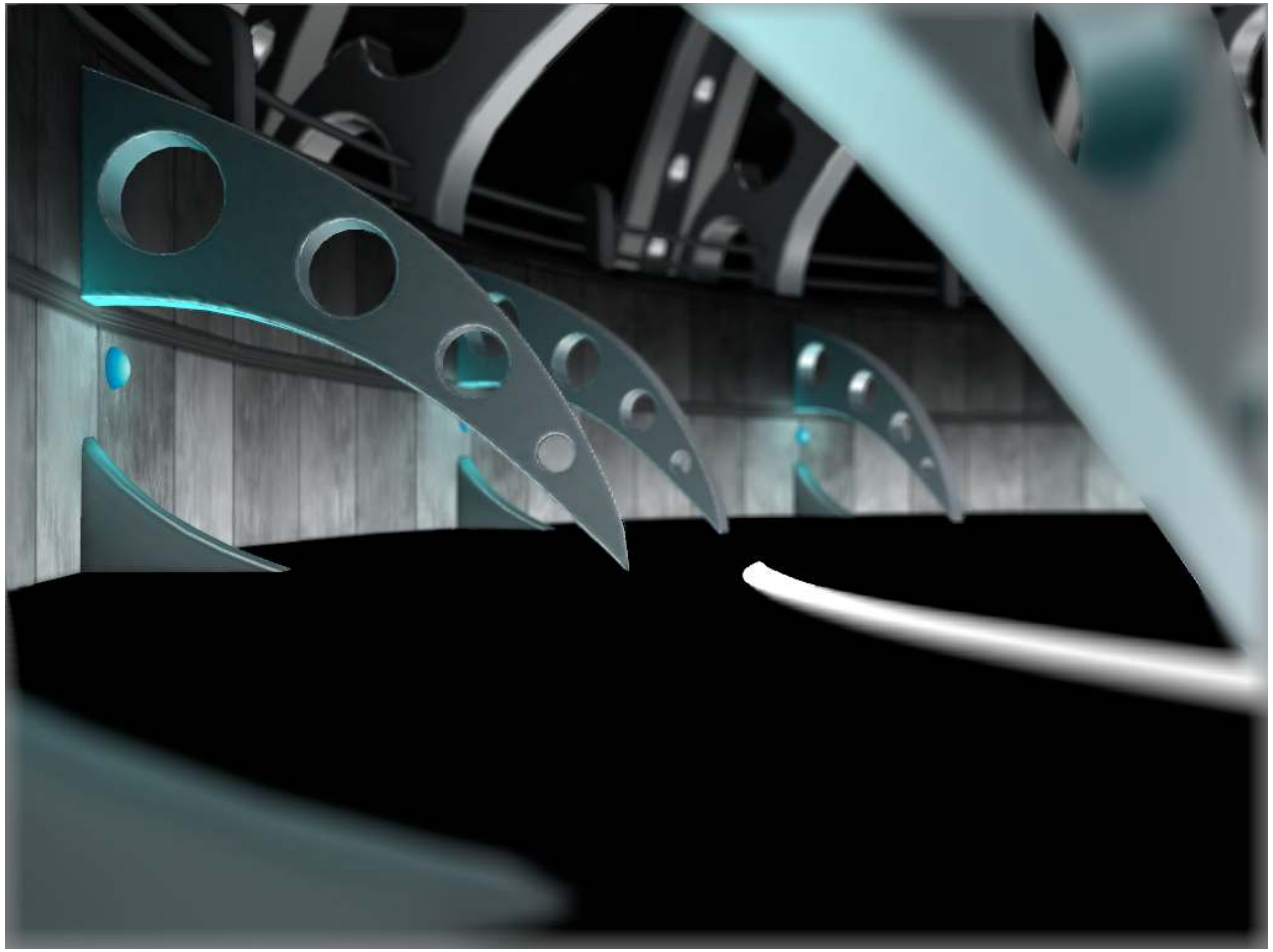
Rendering with Depth-of-Field (Tiefenunschärfe)

1. Render scene, save color buffer and z-buffer (e.g., in texture)
2. Compute summed area table over color buffer
3. For each pixel do *in parallel*:
    1. Read depth of pixel from saved z-buffer
    2. Compute radius of circle of confusion (CoC)
       (for details see "Advanced CG")
    3. Determine size of box filter
    4. Compute average of the pixels within the box
    5. Write in (new) color buffer



- Note: "For each pixel in parallel"could be implemented in OpenGL by rendering a screen-filling quad using special fragment shader

Results

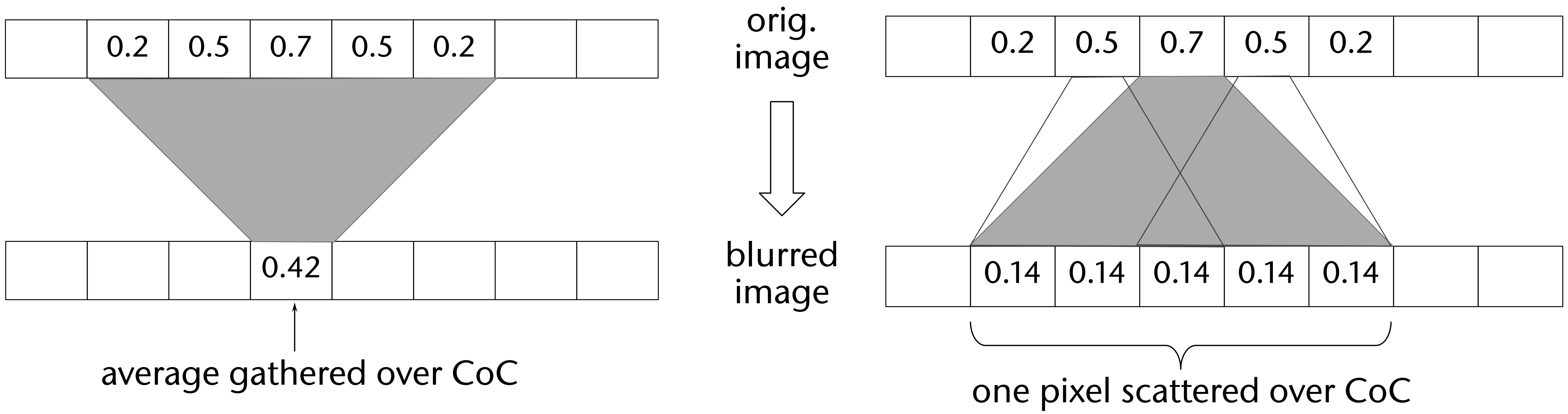


Artifacts of this Technique

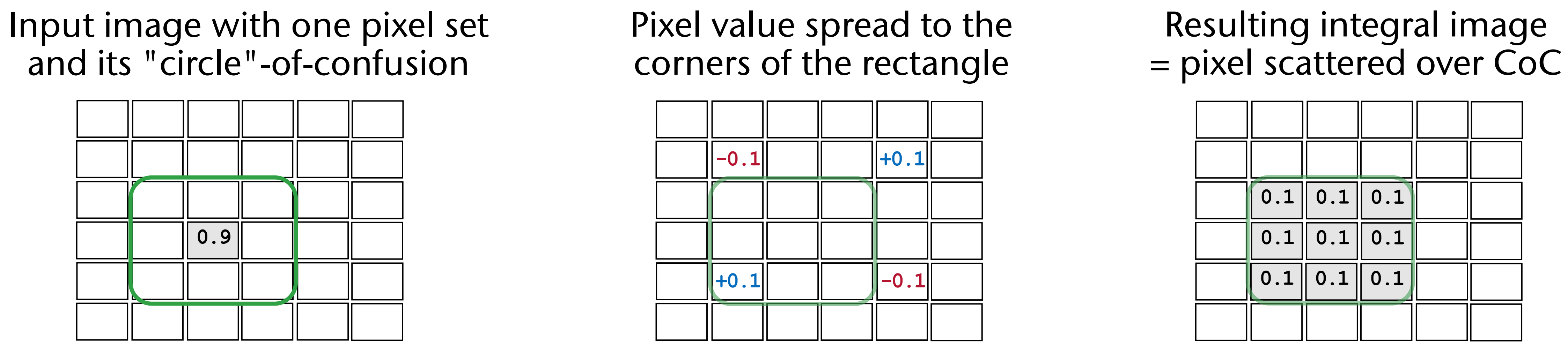
- False sharp silhouettes: blurry objects (out of focus) have sharp silhouette, i.e., won't blur over sharp object (in focus)
- Color bleeding (a.k.a. pixel bleeding): areas in focus can incorrectly bleed into nearby areas out of focus
- Reason: the (indiscriminate) gather operation

Depth-of-Field with Scattering

- Goal: turn gather operation into scatter operation

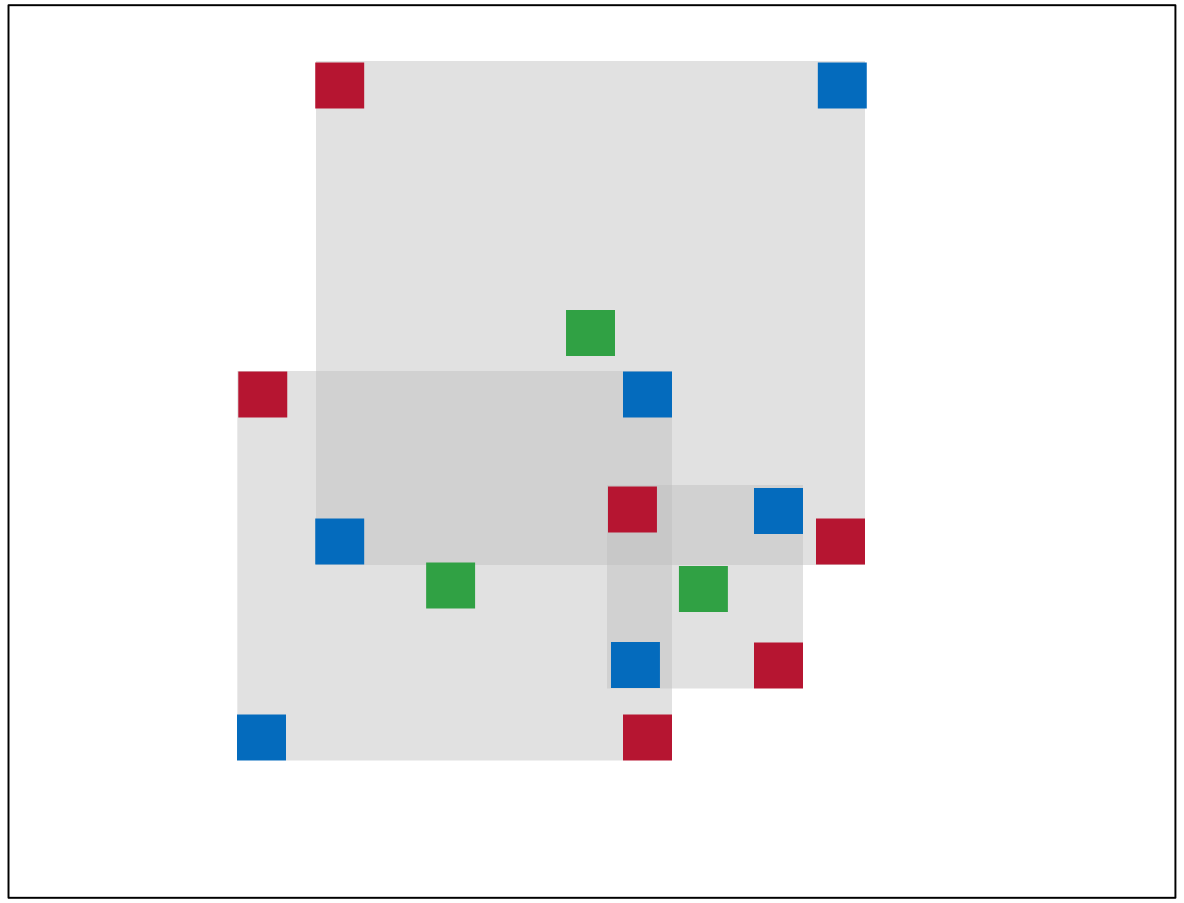


- Example: scatter one pixel using the 2D prefix-sum (integral image)



Algorithm

1. Phase: for each pixel in original image do in parallel:
   - Spread $\frac{\text{pixel value}}{\text{area(CoC)}}$ to CoC corners
     - Use atomic accumulation operation for that!
     - Do this for R, G, and B channels separately
2. Phase: compute 2D prefix-sum over this "scatter image"
   - Result = final image with depth-of-field



- Research question: can you turn phase 1 into a *gather phase*?
  - Would allow to avoid the atomic operations

Result

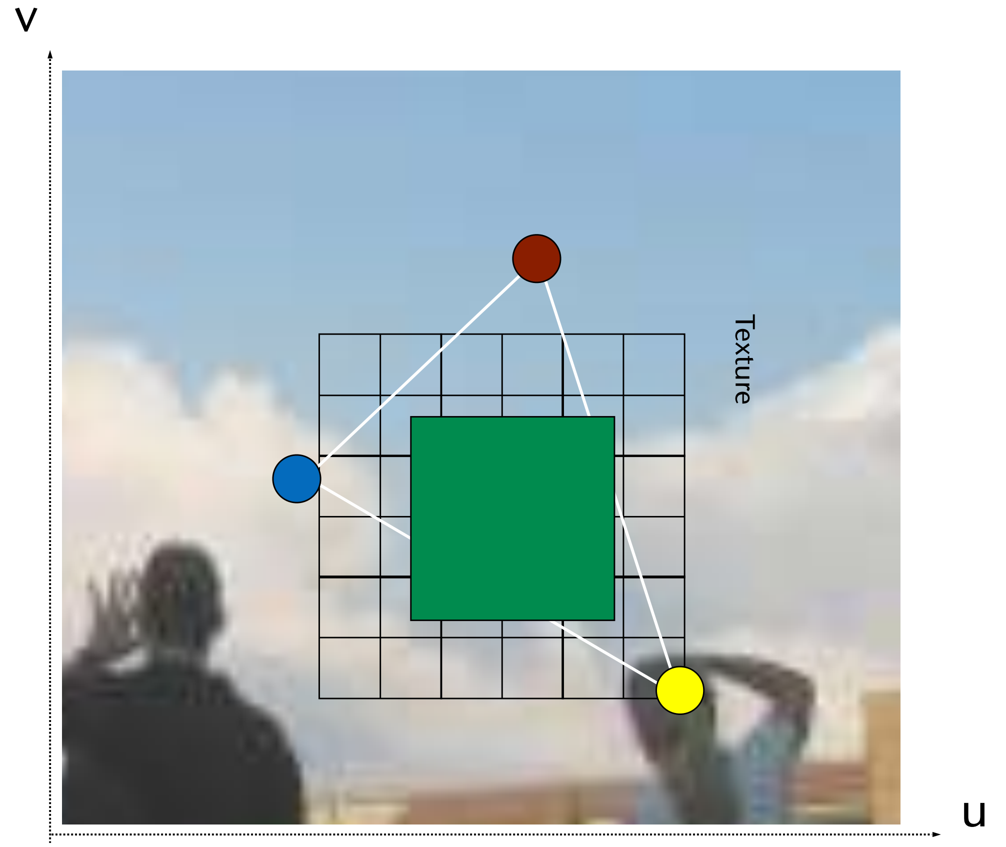
First integral image, then gathering

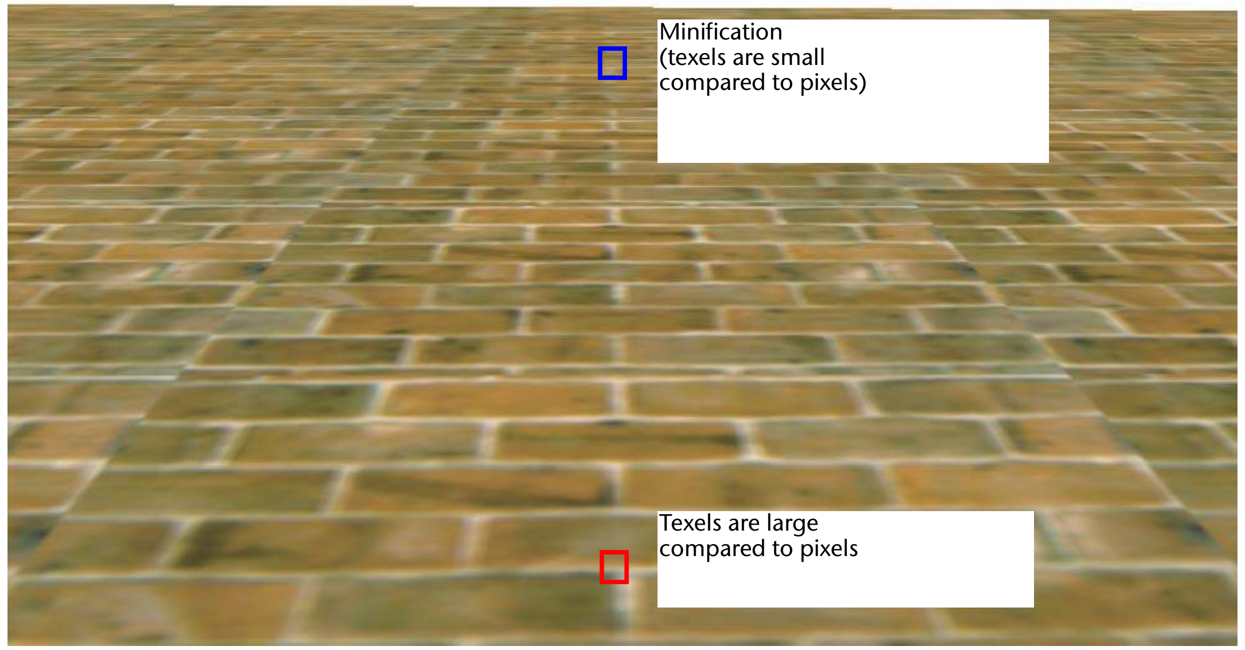
First scattering , then integral image

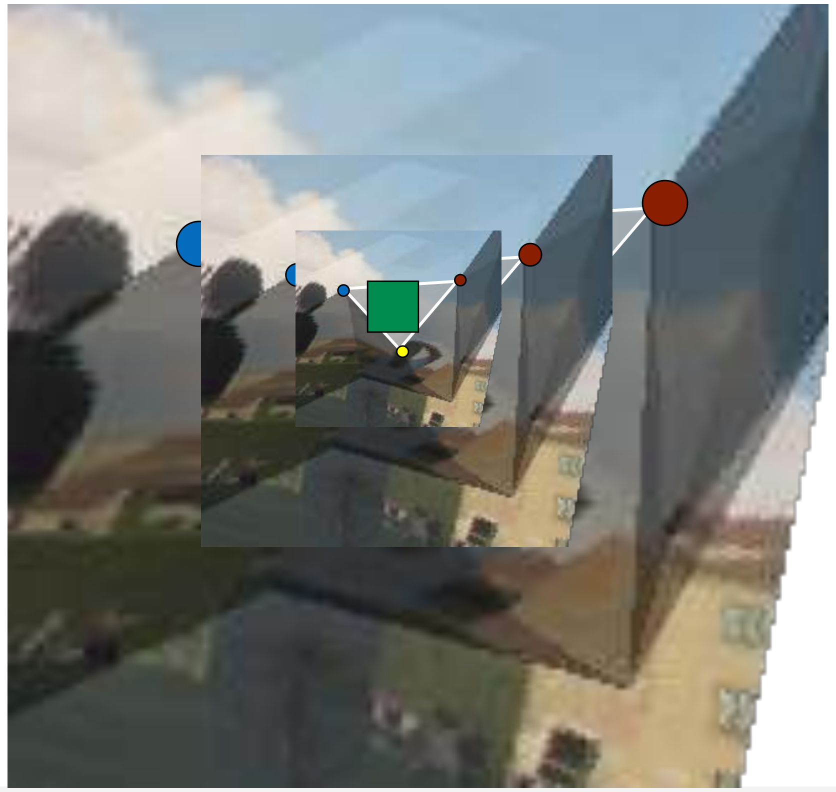
[Kosloff, Tao, Barsky, 2009]

Recap: Texture Filtering in Case of Minification

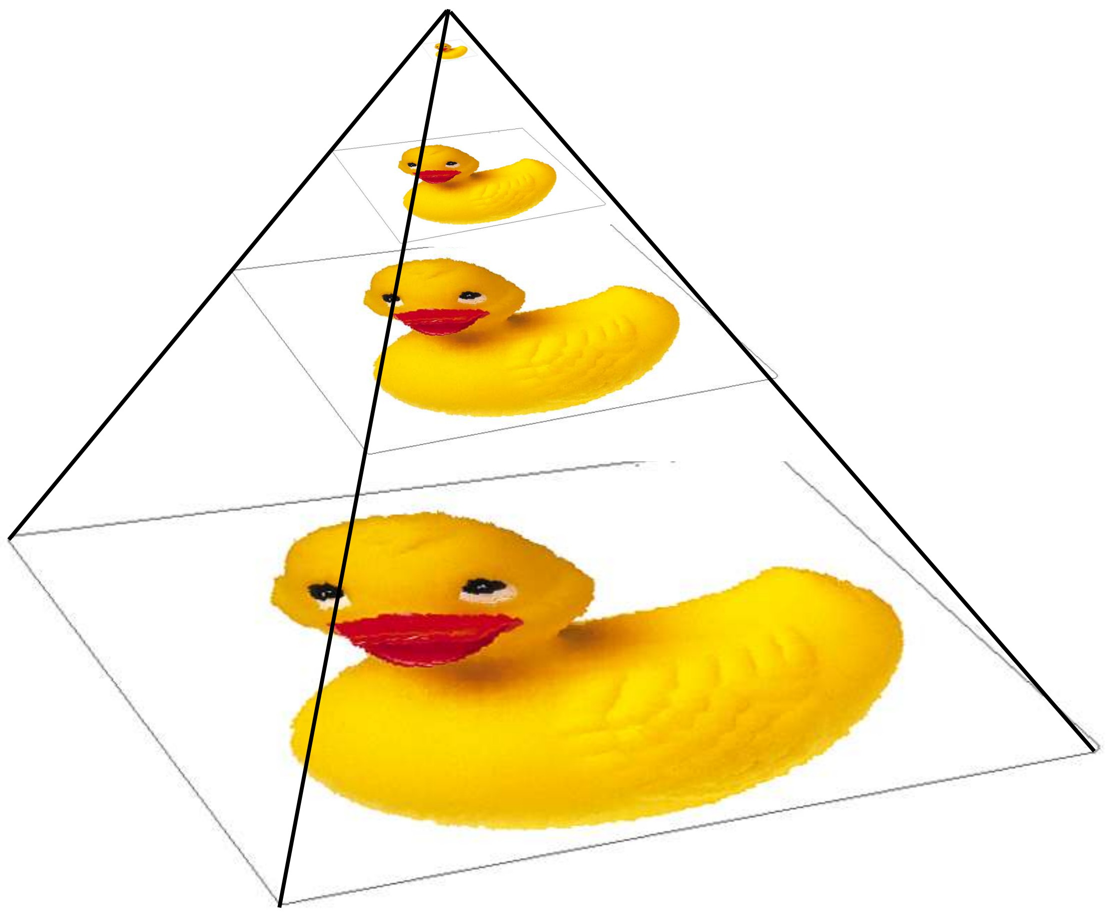
- What happens, when we "zoom away" from the polygon?
- Desired: an averaging of all texels covered by the pixel (in uv-space); too costly at runtime
- Solution: pre-processing → MIP-maps (lat. "multum in parvo" = a lot in a small [space]")

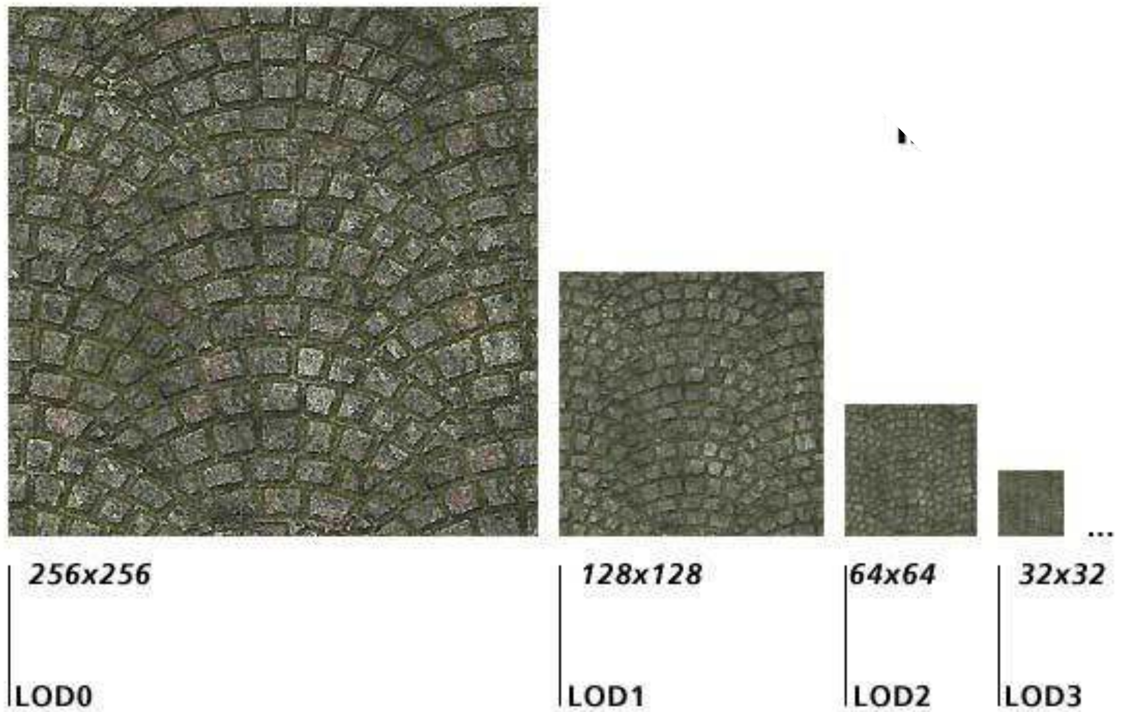






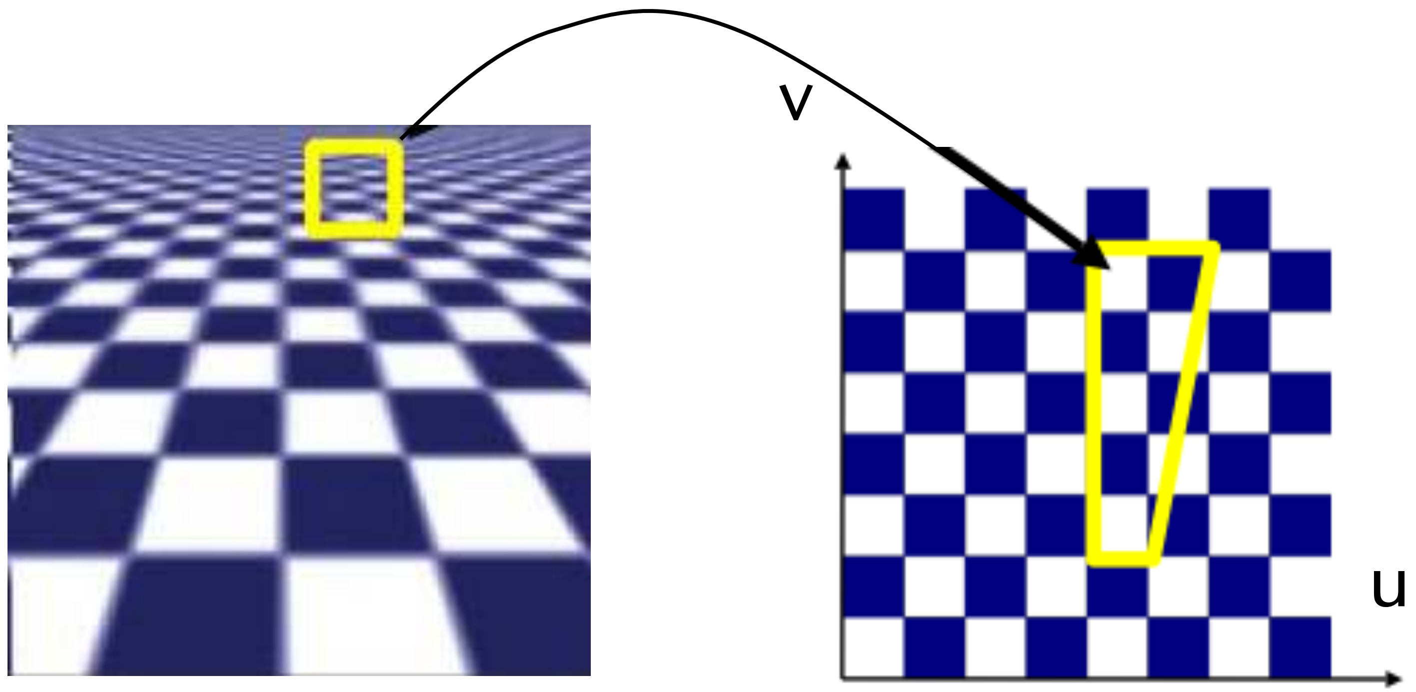
- A MIP-map is just an image pyramid:
  - Each level is obtained by averaging 2x2 pixels of the level below
    - Consequence: the original image must have size $2n$x$2n$ (at least, in practice)
  - You can use more sophisticated ways of filtering, e.g., Gaussian
- Memory usage for MIP-map: 1.3x original size

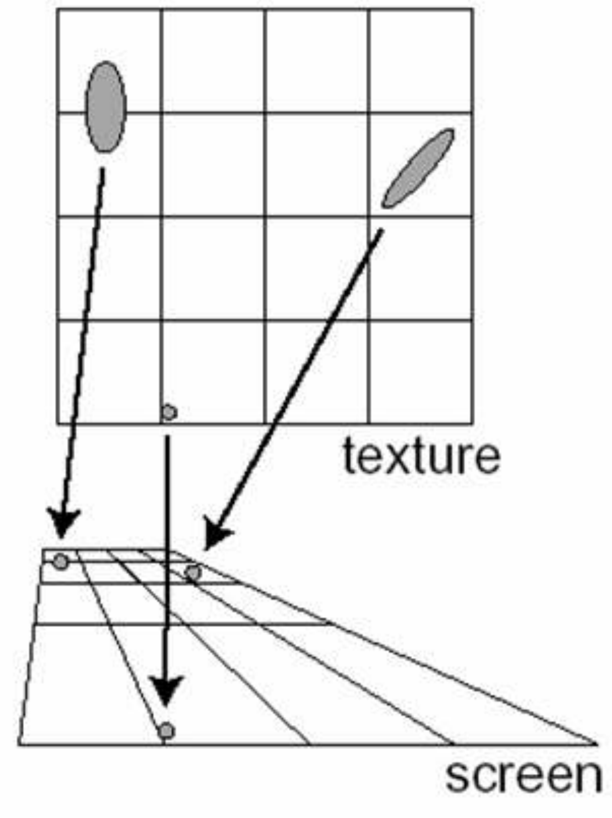




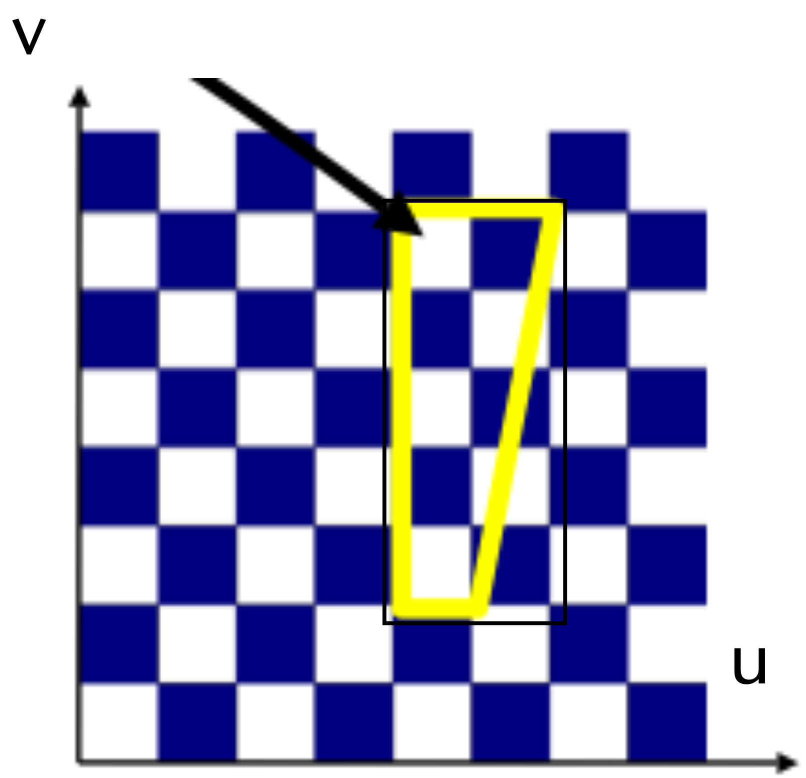
Anisotropic Texture Filtering

- Problem with MIPmapping: doesn't take the "shape" of the pixel in texture space into account!

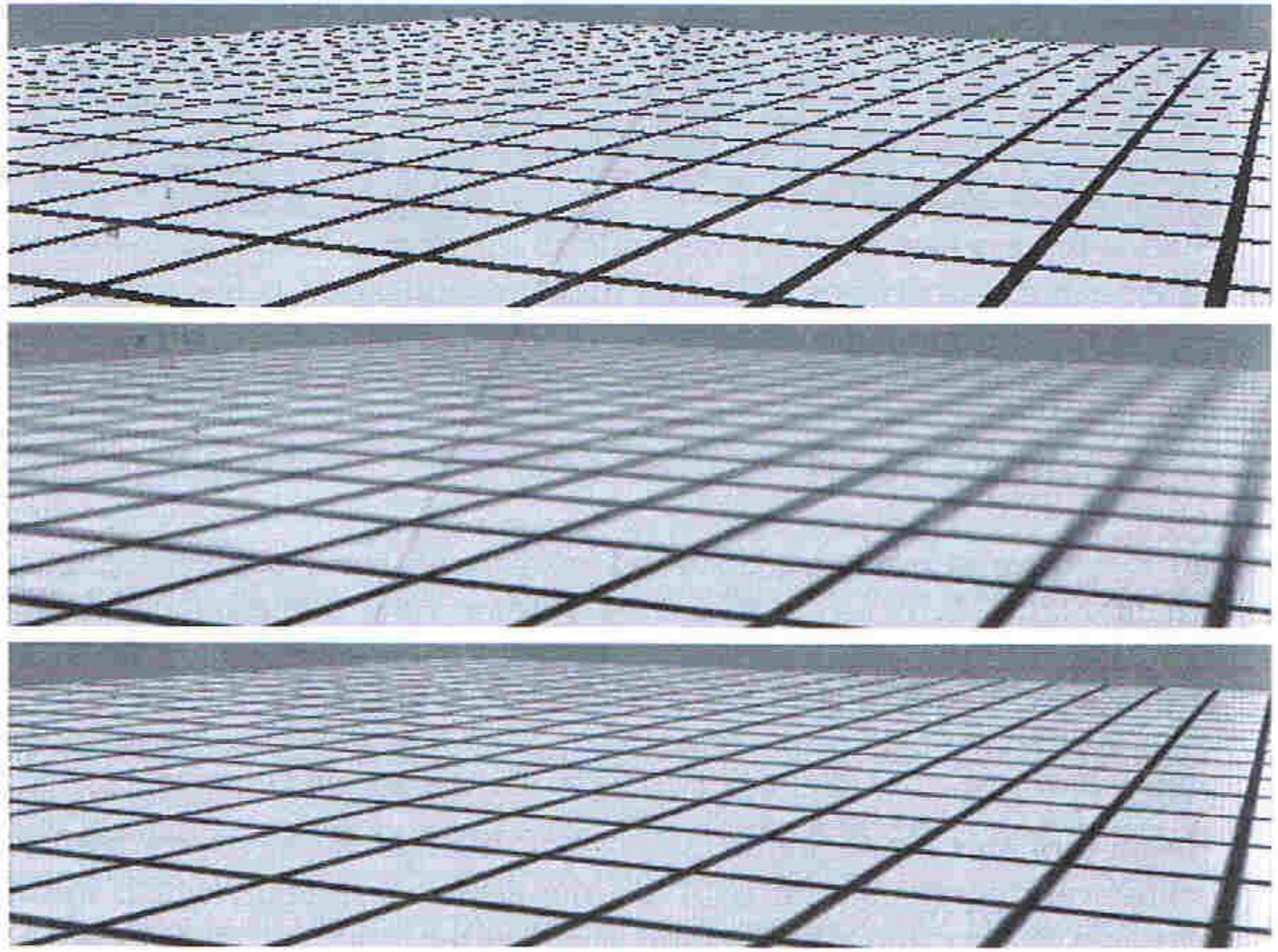




- MIPmapping just puts a square box around the pixel in texture space and averages all texels within
- Solution: average over bounding *rectangle*
  - Use Summed Area Table for quick summation
- Question: how to average over highly "oblique" pixels?



- This is one kind of *anisotropic* texture filtering
- Result:



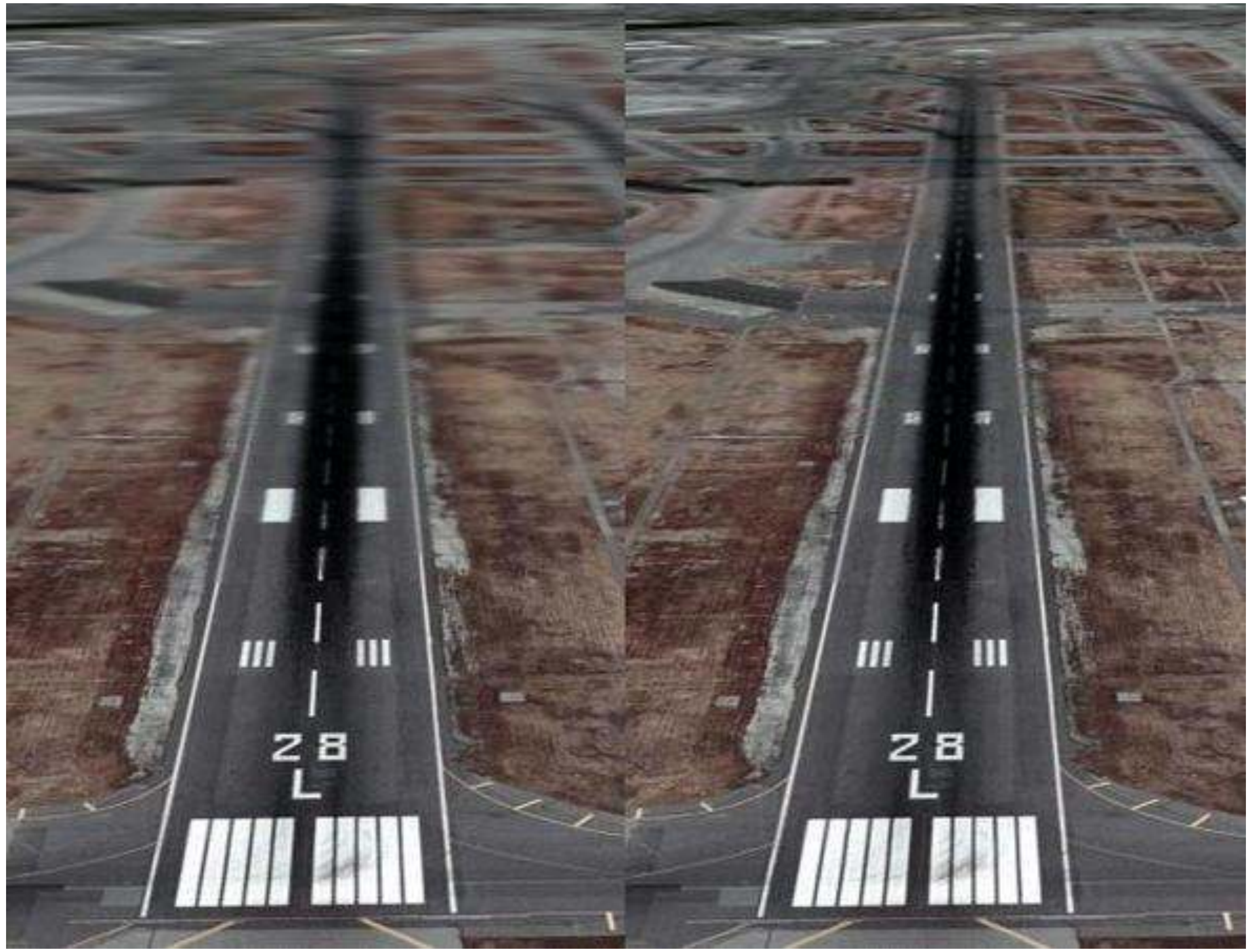
No filtering

Mipmapping

Summed area table

- Another example:

Mipmapping — Anisotropic



- Today: all graphics cards support anisotropic filtering (not necessarily using SATs)

Application: Face Detection

- Goal: detect faces in images (not recognition)

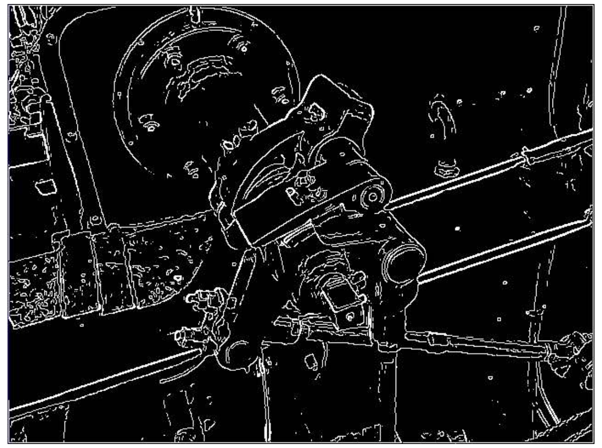
digital camera

iPhoto

Includes a "false positive"
(or does it?)

- Requirements (wishes):
  - Real-time or close (> 2 frames/sec)
  - Robust (high true-positive rate, low false-positive rate)
- Non-goal: face recognition
- In the following: no details, just overview!

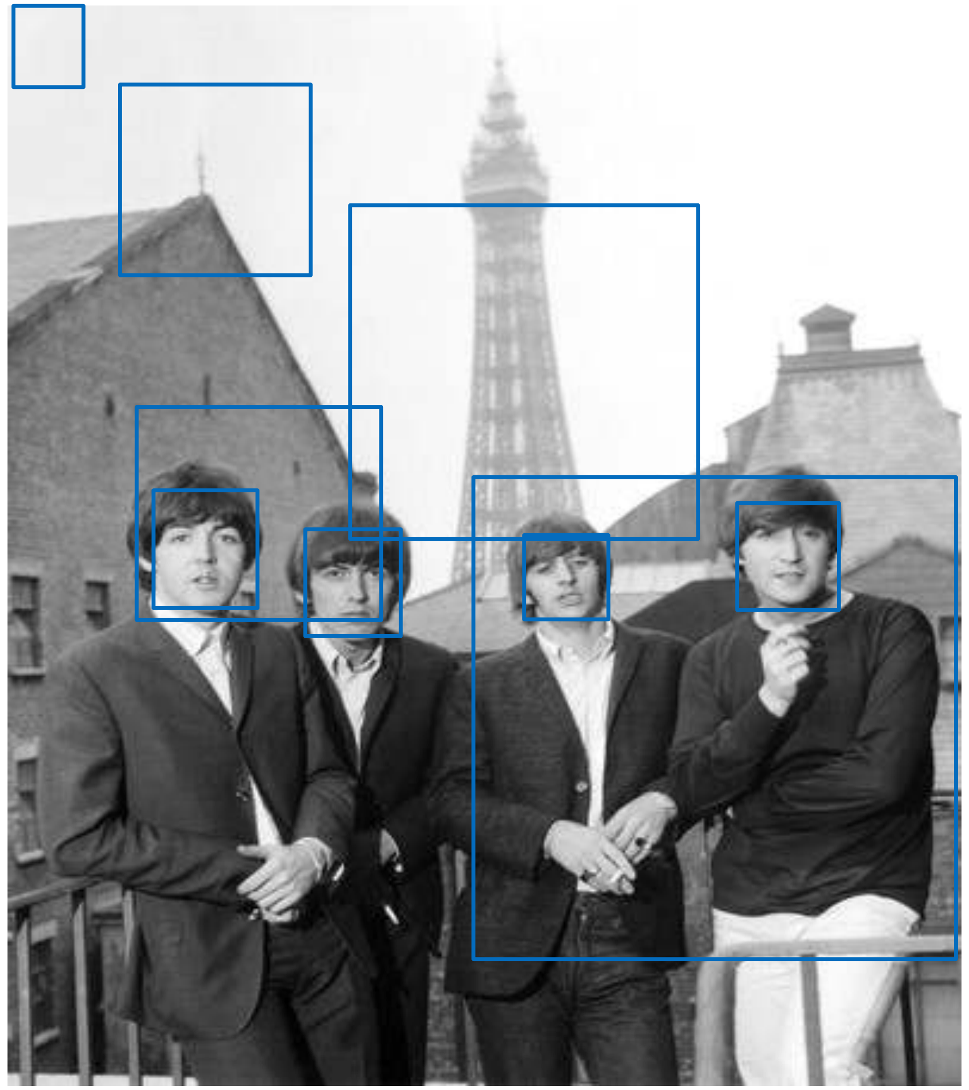
- The term feature in computer vision:
  - Can be literally *any* piece of information/structure present in an image
  - Each kind of feature has a type, each feature has a value
- *Binary features* → present / not present
  - Examples:
    - Edges
    - Color of pixels is within specific range (e.g., skin)



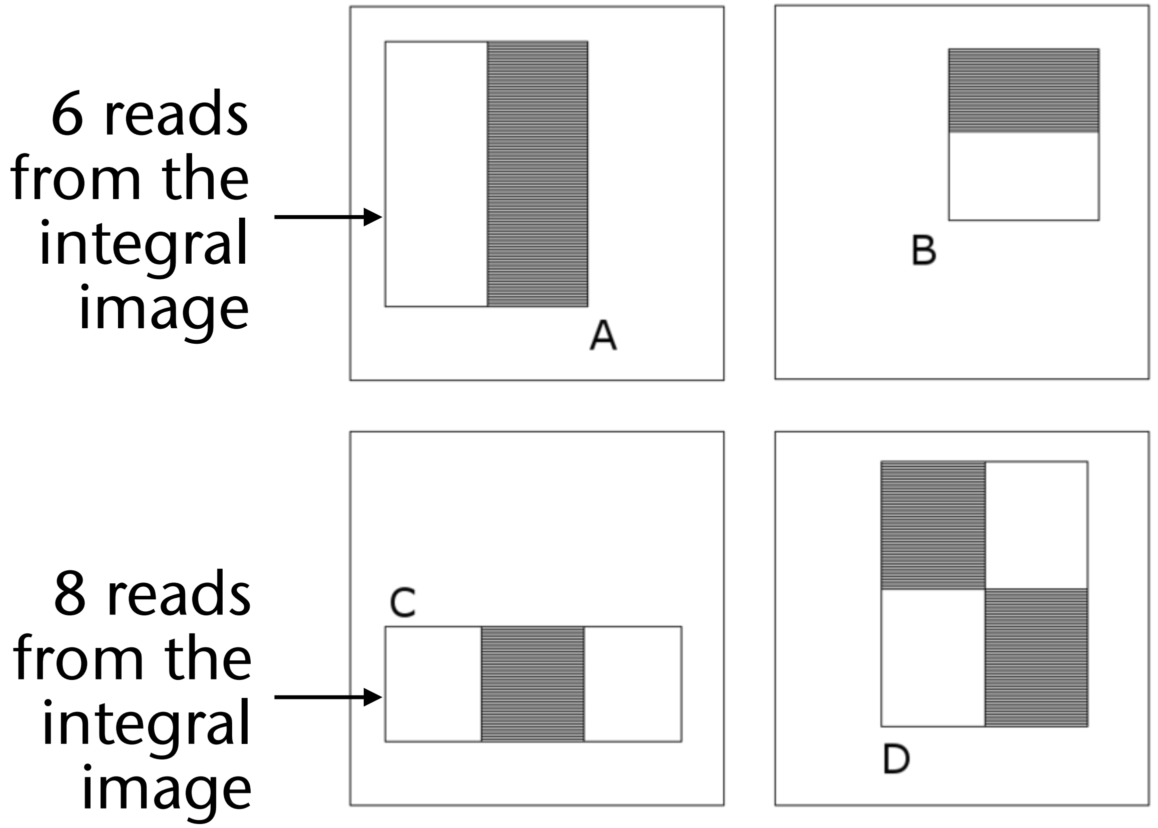
- Non-binary features → probability of occurrence
  - Examples:
    - Gradient image
    - Sum of pixel values within a shape, e.g., rectangle

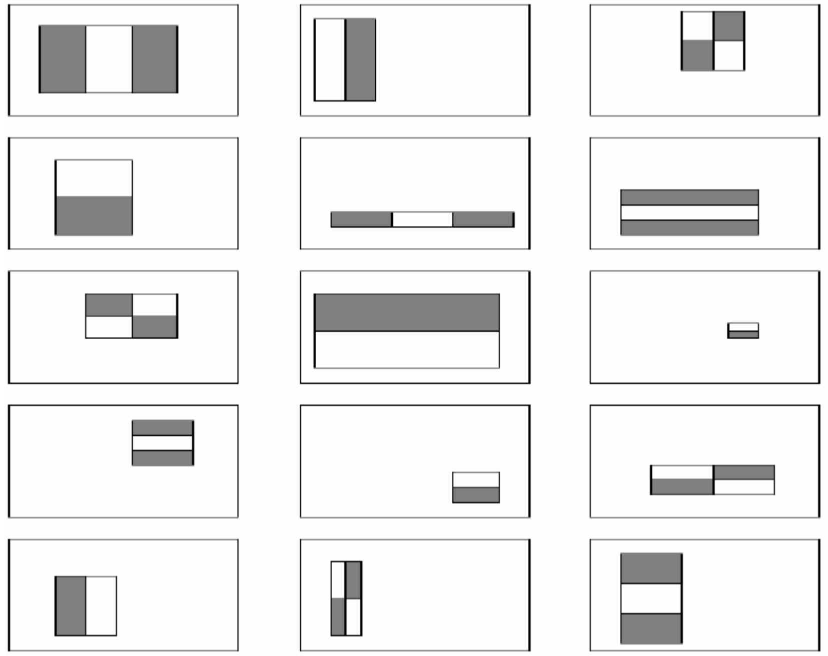
The Viola-Jones Face Detector

- The (simple) idea:
  - Move a sliding window across the image (all possible locations, all possible sizes)
  - Check, whether a face is in the window
  - We are interested only in windows that are filled by a face
- Observation:
  - Image contains 10's of faces
  - But ~10^6 candidate windows
- Consequence: to avoid having a false positive in every image, our false positive rate has to be < 10^{-6}



- Feature types used in the Viola-Jones face detector:
  - 2, 3, or 4 rectangles placed next to each other
  - Called Haar features
- Feature value := g_i = pixel-sum(white rectangle(s)) – pixel-sum(black rectangle(s))
  - Constant time per feature extraction
- In a 24x24 window (e.g., one of the sliding windows), there are ~160,000 possible features
  - All variations of type, size, location within the window

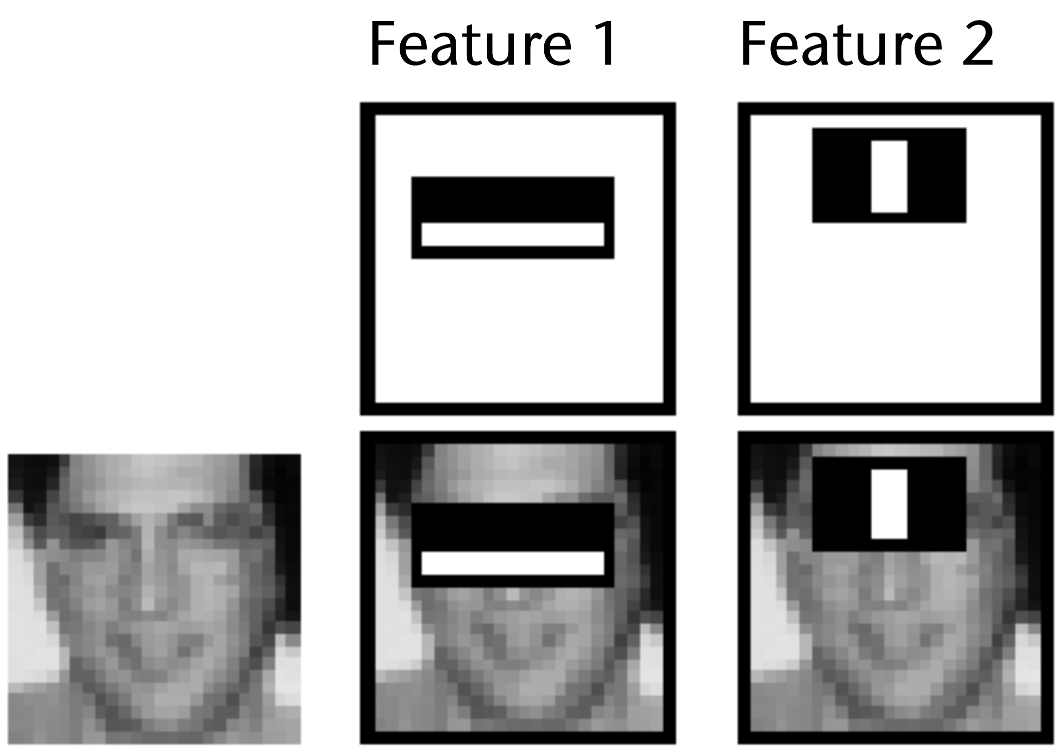




- Define a weak classifier for each feature:

$$f_i = \begin{cases} +1 & , g_i > \theta_i \\ -1 & , \text{else} \end{cases}$$

  - For the two-rectangles feature, for instance, choose $\theta \approx \frac{1}{2} + \varepsilon$
  - Called "weak", because such a classifier is only slightly better than a random "classifier"



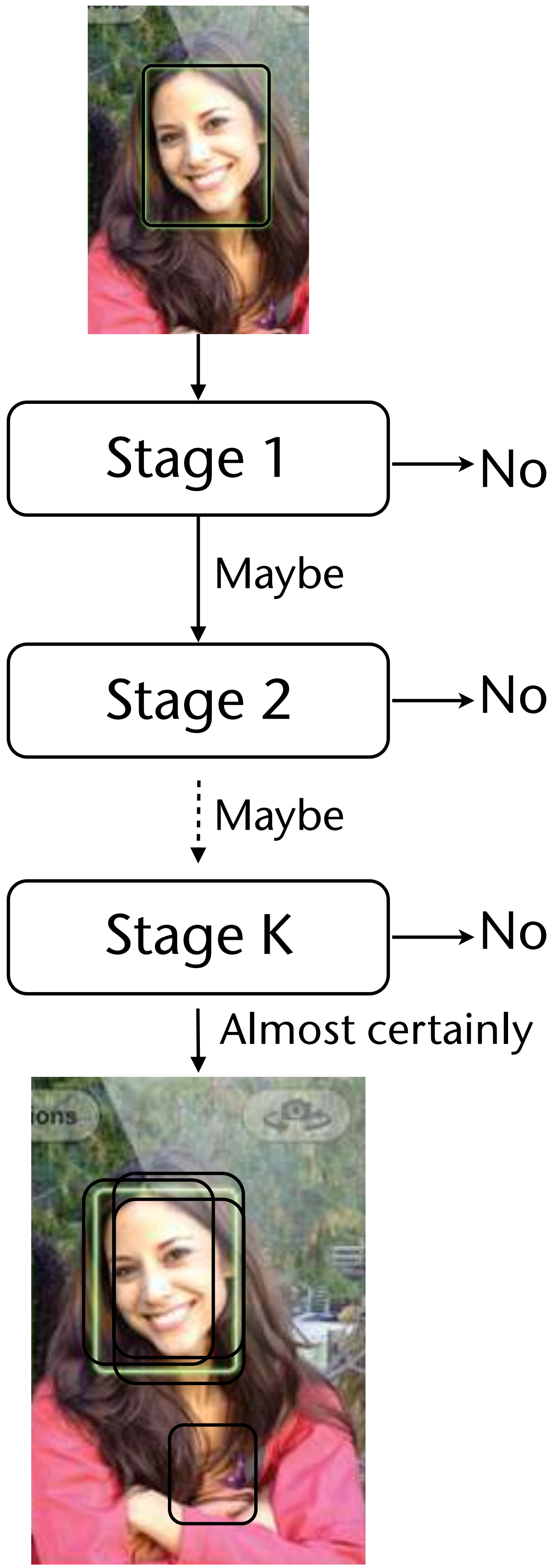
- Idea: combine lots of weak classifiers to form one strong classifier

$$F(\text{window}) = \alpha_1 f_1 + \alpha_2 f_2 + \ldots$$

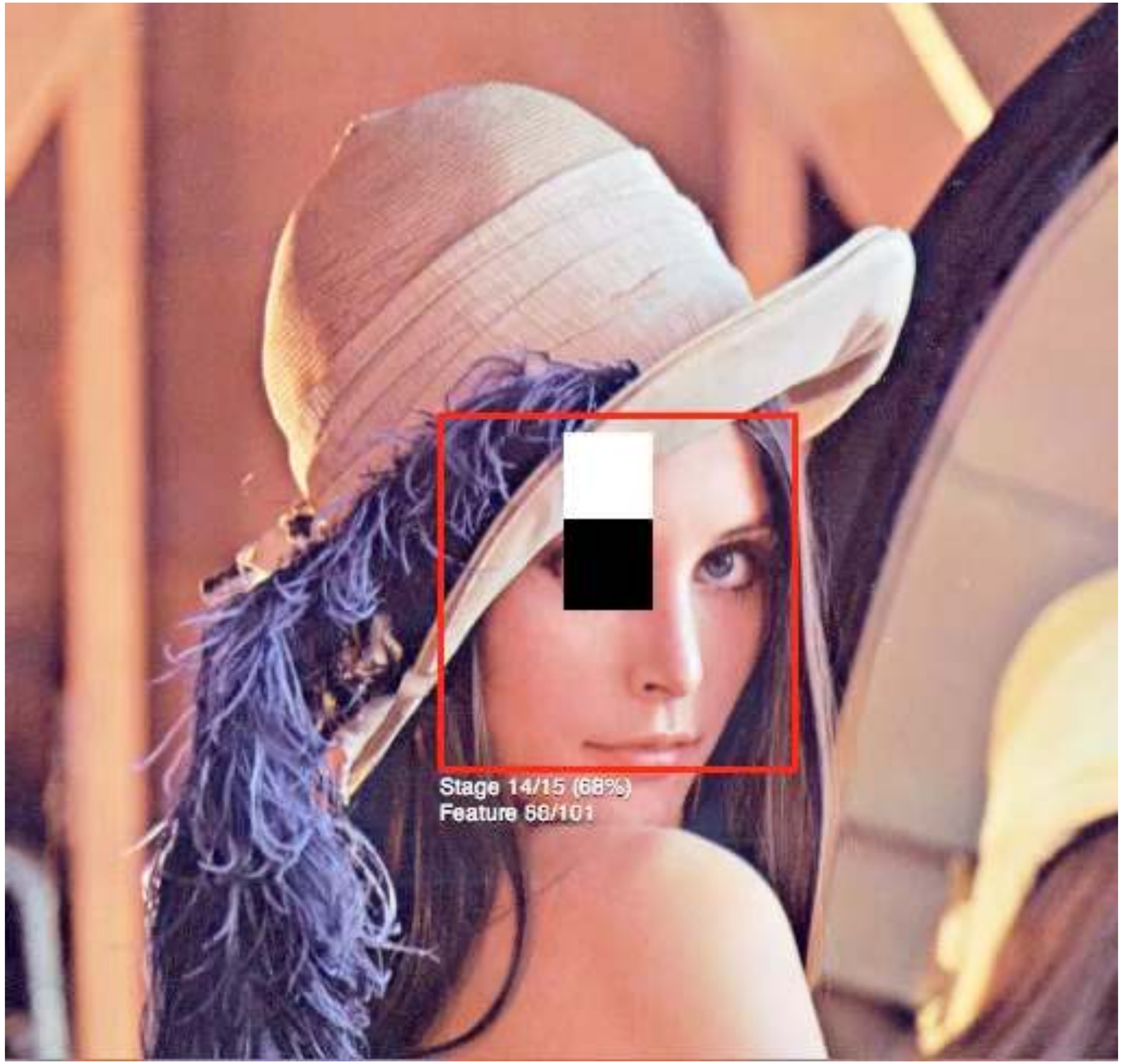
- Use learning algorithms to automatically find a set of weak classifiers and their optimal weights and thresholds, which together form a *strong classifier* (e.g., Random Forest)
  - More on that in AI & machine learning courses
- Training data:
  - 1000's of hand labeled faces containing many variations (illumination, pose, skin color, ...)
  - 10000 non-faces
- Weak classifiers with largest weights are meaningful and have high discriminative power (use first *k* of them):
  - Eyes region is darker than the upper-cheeks
  - Nose bridge region is brighter than the eyes

Some Details on Optimizations

- Arrange in a filter cascade:
  - *K* classifiers with highest weights come first
  - If window fails one a stage in the cascade → discard window
    - Advantage: "early exit" if "clearly" non-face
  - Typical detector has 38 stages in the cascade, ~6000 features/weak classifiers
- Final stage: only report face, if cascade finds several nearby face windows
  - Discard "lonesome" windows



Visualization of the Algorithm



Adam Harv
(http://vimeo.com/12774628)

Final remarks on Viola-Jones

- Pros:
  - Extremely fast feature computation
  - Scale and location invariant detector
    - Instead of scaling the image itself (e.g. pyramid-filters), we scale the features
  - Works also for some other types of objects
- Cons:
  - Doesn't work very well for 45° views on faces
  - Not rotation invariant