Voronoi diagrams

Notation: $d(p,q) = ||p-q||$: distance between 2 points

$\bar{R}$= closure of the region $\underline{R}$

Bisector: Let p, q be points, $B(p,q) = \{x | d(x,p) = d(x,q)\}$, the locus of points equidistant from p and q

Voronoi diagra,

We will define:

$H(p,q) = \{x | d(x,p) < d(x,q)\}$

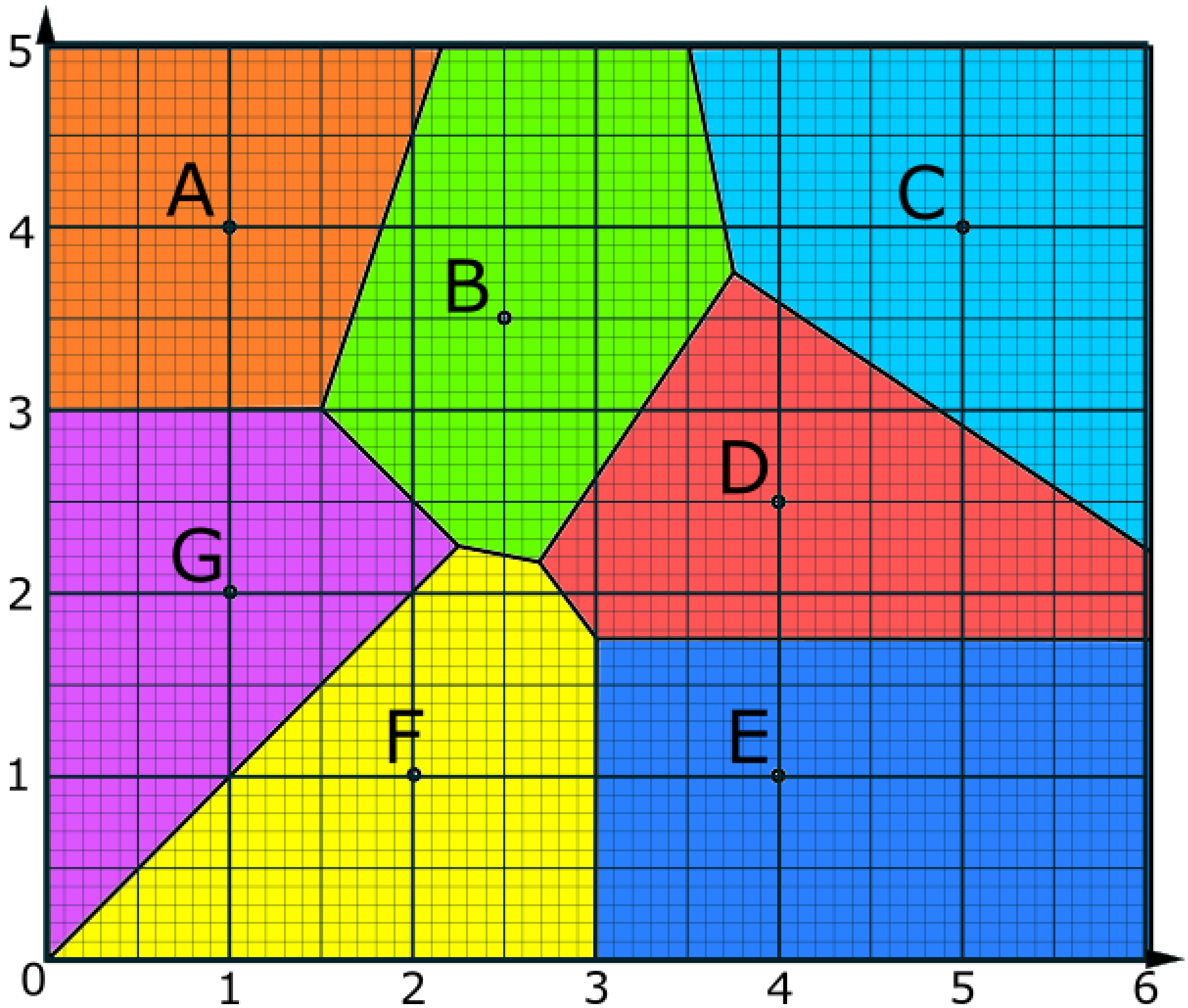
$H(q,p) = \{x | d(x,p) > d(x,q)\}$

given a set S of points , let $p \in S$, the **voronoi region** of p is the interception of convex halfspaces

$$R(p) = \bigcap_{p_i \in S \setminus p} H(p_i, p)$$

We define the voronoi diagram of S

$$V(s) = \bigcup_{p,q \in S, p \neq s} \bar{R}(p) \cap \bar{R}(q)$$



Properties

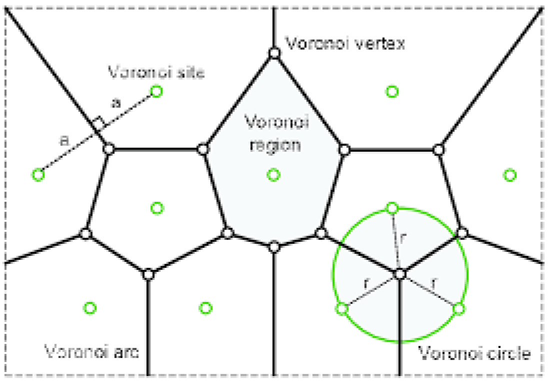
$\forall p \in S, R(p)$ is convex

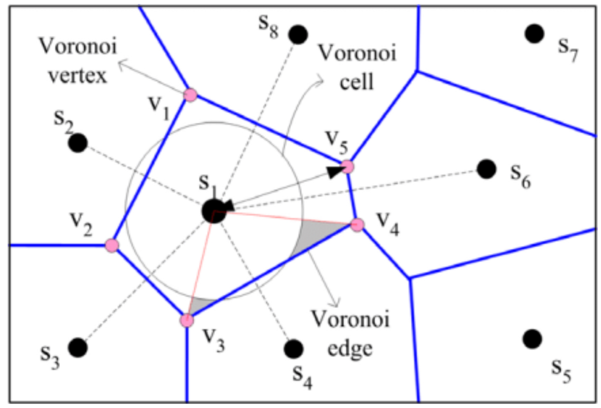
$\forall p, q \in S, p \neq q, R(p) \cap R(q) = \emptyset$

$\bar{R}(p) \cap \bar{R}(q) \subseteq \bar{H}(p,q) \cap \bar{H}(q,p) = B(p,q)$

At most one contiguous piece of $B(p,q)$is part of $\bar{R}(p)$

$R(p)$ is the set of all the points closer to p than any other points in S





Theorem of the expanding Sphere

Let S be a set of points, let $x \in S$. Consider a sphere(circle in 2D) around x and consider it expanding.

At some point $C(x)$ will touch a one or more points in S

Cases:

- exactly one point $p \in S$:$x \in R(p)$
- exactly two point $p, q \in S$:$x \in$ region between $R(p), R(q)$
- Three or more points $p_1, ..., p_k \in S, x =$ voronoi node between $R(p_1), ..., R(p_k)$

Proof

//Todo

Lemma

A point x s on a **voronoi edge** $\iff \exists C(x)$ that touches exactly 2 points in S and contains no other points from S

A point x s on a **voronoi node** $\iff \exists C(x)$ that touches three or more points in S and contains no other points from S

Global properties

Lemma: There is a connection between $V(S)$ and $CH(S)$: $R(p)$ is unbounded $\iff$ p is on the border of $CH(S)$

Proof: $R(p)$ is unbounded, let e=unbounded edge of $R(n)$, q= site s.t. $E \subseteq B(p,q)$. Consider $C(x)$ through p,q and x on e.

Let $\alpha \to \infty \Rightarrow$segment of $C(x)$ between p,q approaches $\bar{pq}$ but $C(x)$ never contains any other $r \in S \Leftarrow$ is neighbor of $q \Rightarrow \exists C(x)$ through p,q containing no other $r \in S$, so we can expand $C(x)$

Note: this proof is valid only in general positions(depending on the context), in this case ni three points are on $CH(S)$ are on the same straight line

Complexity

a voronoi over n points in the plane has $O(n)$ nodes, edges and regions

Proof:

Pruning infinite edges from a voronoi by a circle that is *big enough* to contain all the points, replacing this circle segments by straight lines. Consider the voronoi diagram plus the new edges as a polyedron developed into the plane, evert $R(p)$ corresponds to one face, infinite regions correspond to the last face

Euler's equation applies to voronoi, so the complexity claims can be transferred.

Lemma:the construction of $V(s)$ in the plane takes $\Omega(n \log n)$

Proof: reduce the problem of constructing a CH to a voronoi, so we can derive the CH from the voronoi in time $O(n)$