Mathematics and Colour

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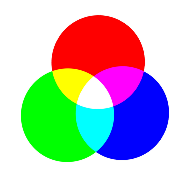
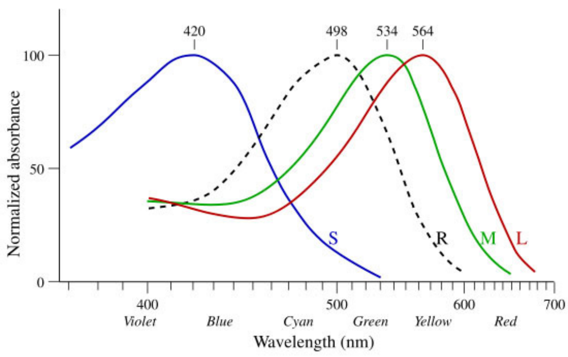
`http://www.manchester.ac.uk/~higham/`



What is Colour?

- Human retina has 3 types of cones.



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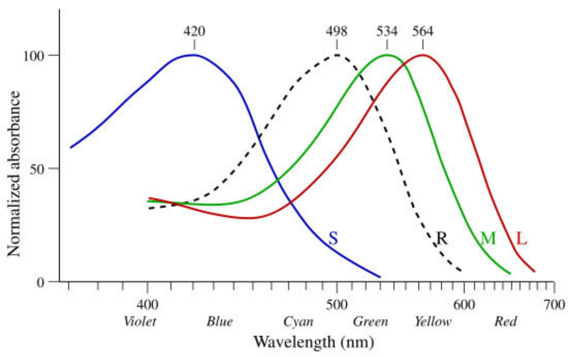
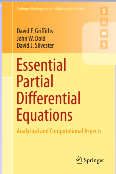
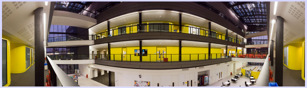


- Colour space is 3-dimensional (“trichromatic theory”).
- Can mathematics help us understand colour?

There’s Something about Yellow





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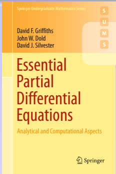
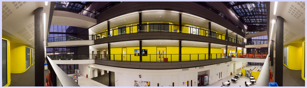
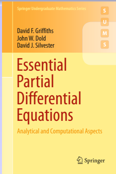


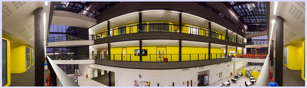


- Why does yellow appear so bright?

There's Something about Yellow







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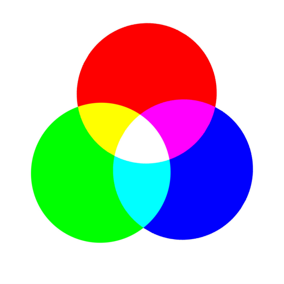
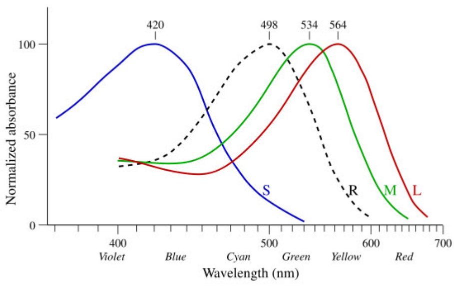
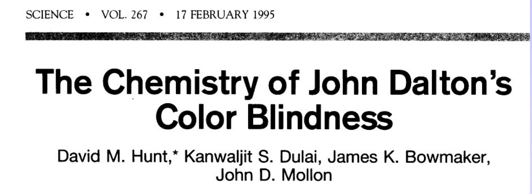



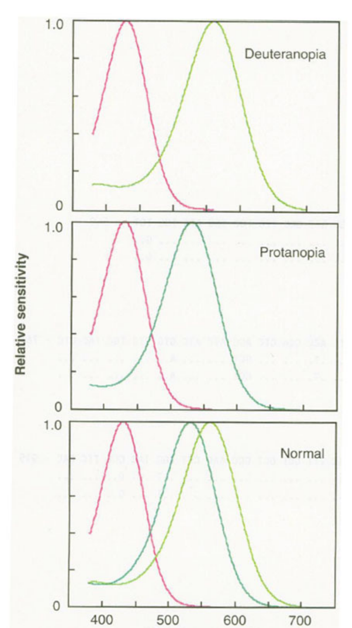
Colour Blindness


SCIENCE • VOL. 267 • 17 FEBRUARY 1995

The Chemistry of John Dalton's Color Blindness

David M. Hunt,* Kanwaljit S. Dulai, James K. Bowmaker, John D. Mollon

- John Dalton (1766–1844).
- Described his own c.b. in lecture to M/cr Lit & Phil Soc, 1794.
- He was a deuteranope.



Vector Space Model of Colour (1)

- Model responses of the 3 cones as

$$c_i = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} s_i(\lambda) f(\lambda) d\lambda, \quad i = 1\colon 3,$$

where $f =$ spectral distrib. of light, $s_i =$ sensitivity of ith cone, $[\lambda_{\text{min}}, \lambda_{\text{max}}] =$ wavelengths of visible spectrum.

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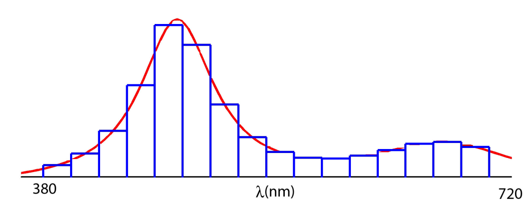
- **Discretizing** gives

$$c = S^T f, \qquad c \in \mathbb{R}^3, \quad S \in \mathbb{R}^{n\times 3}, \quad f \in \mathbb{R}^n.$$

For standardized S, c is the **tristimulus** vector.



Vector Space Model of Colour (2)

- Let columns of $P = \underbrace{[p_1\ p_2\ p_3]}_{n\times 3}$ be **colour primaries**.
- Assuming $S^T P$ is nonsingular,

$$S^T f = \underbrace{S^T P}_{3\times 3} \cdot (S^T P)^{-1} S^T f \equiv S^T \cdot P a(f),$$

where $a(f) = (S^T P)^{-1} S^T f$.
Colour of any spectrum f can be matched by primaries.

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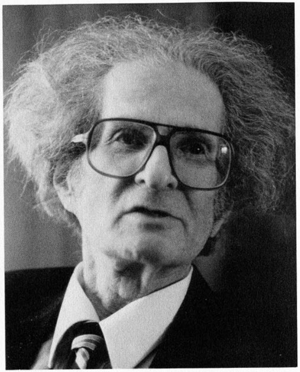
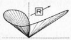
Colour of any spectrum f can be matched by primaries.

- Need $a_i \geq 0 \Rightarrow$ not all visible spectra can be produced. Compensate $a_i < 0$ by adding $|a_i| p_i$ to f,
- There exist spectra f, g, $f \neq g$, such that $\underbrace{S^T}_{3\times n} f = S^T g$: **metamers**. Both good and bad.



Jozef B. Cohen, 1921–1995 (Photo by Jerry Thompson)

VISUAL COLOR AND COLOR MIXTURE



The Fundamental Color Space

Jozef B. Cohen

UNIVERSITY OF ILLINOIS PRESS
URBANA AND CHICAGO

R Matrix Theory of Cohen

Cohen (2001) stresses the importance of

$$R = S(S^T S)^{-1} S^T = SS^+,$$

the orthogonal projector on range(S).

- Independent of the choice of primaries used for colour matching ($S \leftarrow SZ$).
- F matrix defined as Q in the factorization $S = QL$ ($Q \in \mathbb{R}^{n\times 3}$, $Q^T Q = I$, $L \in \mathbb{R}^{3\times 3}$ lower triangular).
- Proposes use of *tricolor coordinates* $F^T f$.

A Nonlinear, Imperfect World

Limitations on how far the mathematical model can take us.

- We all see colour slightly **differently.**
- Our eyes do not behave **linearly**.
- Brain processing of colour is complicated (**colour temp**, **opponent-process theory**) and leads to various illusions.
- Most colours we see are **artificially generated**: camera, screen, print, paints, . . . *all these devices have limitations*.

CMYK

All printing is done using four colours: **cyan**, **yellow**, **magenta**, and **black**.

$C + M + Y = K = \text{black}$.

One *redundant coordinate*. Why do we need K?

CMYK

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One *redundant coordinate*. Why do we need K?

- Printing 3 layers makes the paper very wet.
- Black as 3 layers requires accurate registration.
- $C + M + Y$ will not give a true, deep black due to ink imperfections.
- Coloured ink is more expensive.

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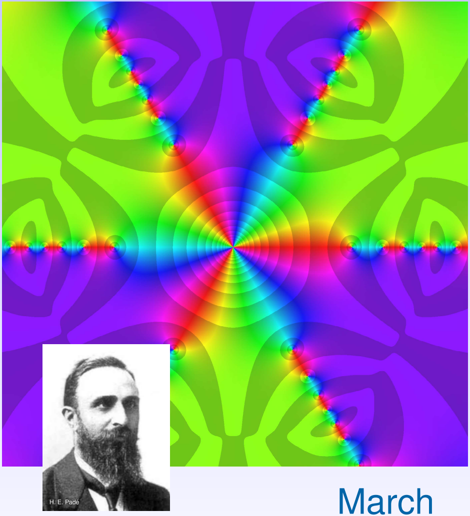
What order to lay down the inks?

CMYK or KCMY are standard. Note that

$$C + M + Y \neq M + C + Y.$$

Complex Beauties Calendar [link]

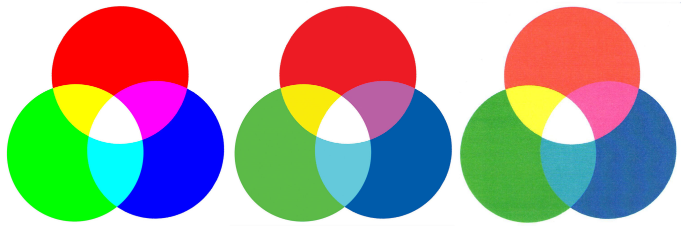


March

Su	Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa
			1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22	23	24	25
26	27	28	29	30	31								

CMYK vs RGB

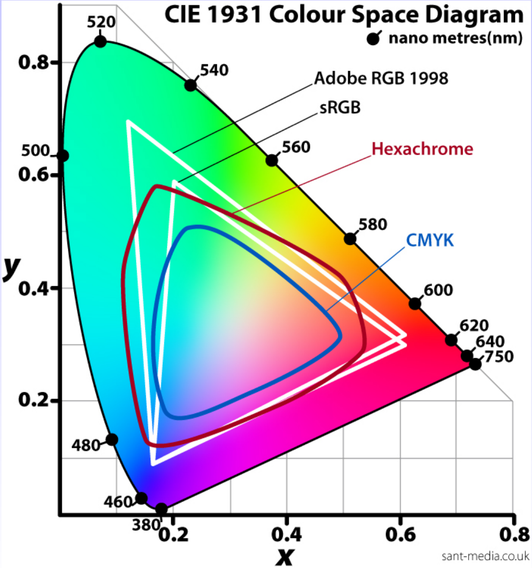
- CMYK produces a different range of colors than RGB. Cannot produce some of the brilliant blues.
- Whenever we print a document on a laser printer we view a CMYK representation of the colors.



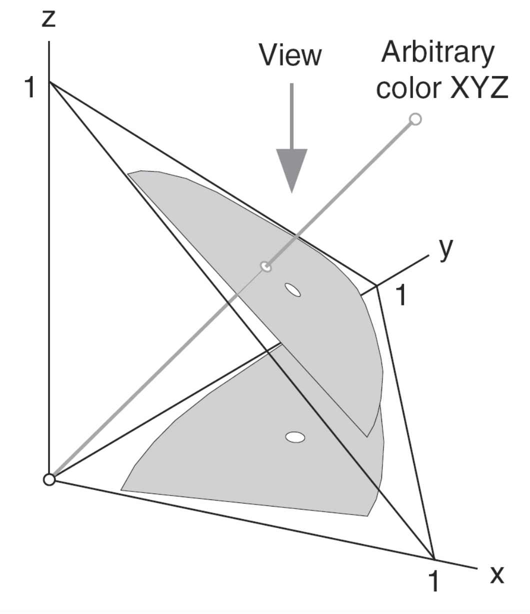
Original Print Scan of print

CIE Chromacity Coordinates

Projective transformation of 3-dimensional colour space.



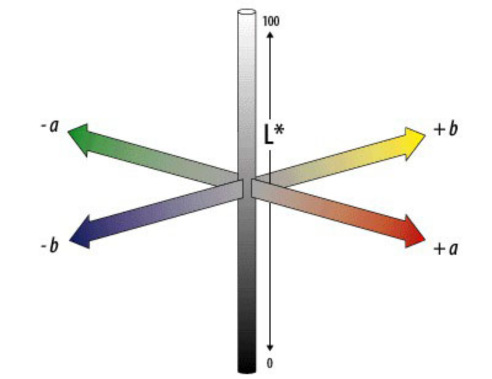
Projective Transformation



Excursion into LAB Space

Change from RGB space to **CIE L*a*b*** (**LAB**, 1976):
L = lightness, **A** = green–magenta, **B** = blue–yellow.

- Separates luminosity from colour.
- More perceptually uniform.



Transformation XYZ $\to$ LAB

Let X_n, Y_n, Z_n be tristimuli of white stimulus.

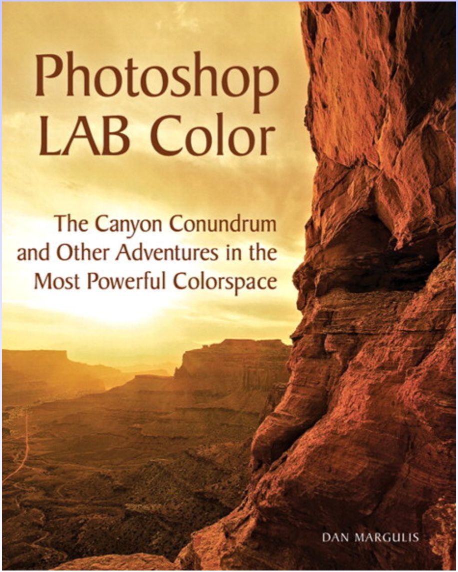
$$
\begin{aligned}
L &= 116 f(Y/Y_n) - 16, \\
A &= 500\left[f(X/X_n) - f(Y/Y_n)\right], \\
B &= 200\left[f(Y/Y_n) - f(Z/Z_n)\right].
\end{aligned}
$$

where

$$
f(x) = \begin{cases} x^{1/3}, & x \ge 0.008856, \\ 7.787x + \frac{16}{116}, & x \le 0.008856. \end{cases}
$$

- Range: $0 \le L \le 100$.
- $A = B = 0 \Rightarrow$ no colour.
- Euclidean distance used as colour difference metric.

Dan Margulis on LAB (2006)



Editing in LAB

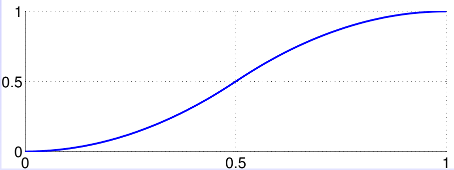
- LAB separates luminosity (L) from colour (A,B).
- Colour noise can be handled by blurring the A, B channels.
- Much bigger space than sRGB with many **imaginary** colours.
- Good for boosting contrast, enhancing colours, and sharpening.

LAB Example: Original

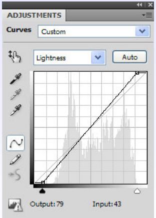
LAB Example: Via LAB

LAB Example: Explanation

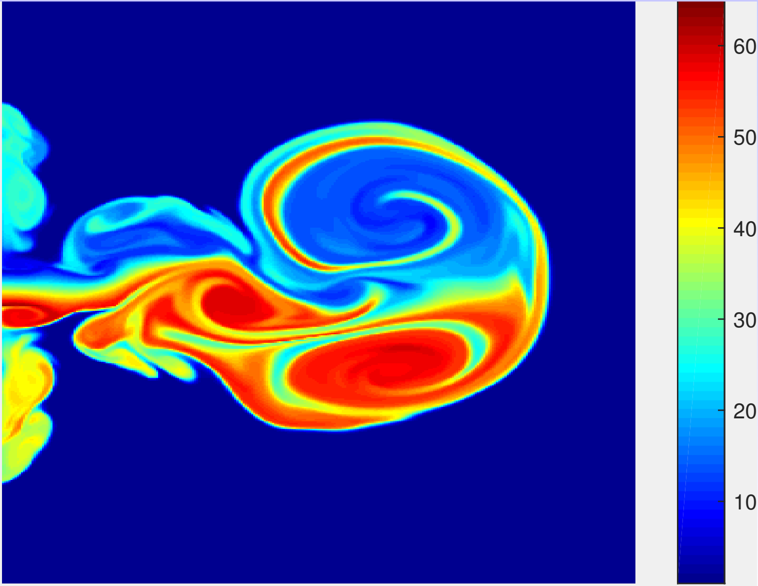
1. Convert from RGB to LAB.
2. **Apply Image** to itself in overlay mode:
   $L \leftarrow f(L)$, $A \leftarrow f(A)$, $B \leftarrow f(B)$, where



3. **Apply Image**: L $\leftarrow$ 75% old L + 25% new L.
4. Curves adjustment on L channel:

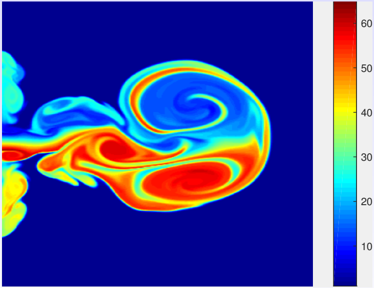


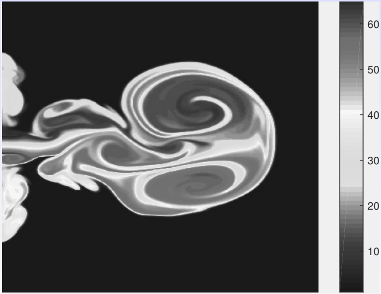
Rainbow Colour Maps



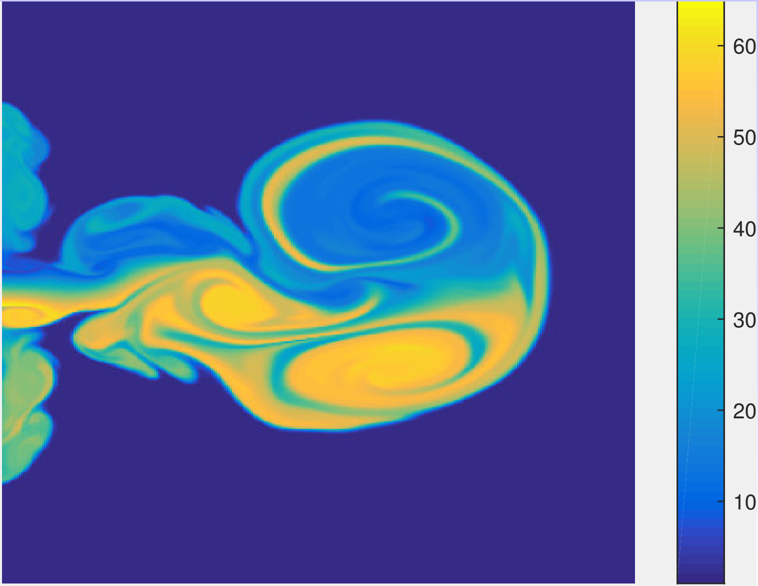
Rainbow Considered Harmful

- Not perceptually uniform: colours change at different rates.
- Confusing: no natural ordering (ROYGBIV).
- Introduces artefacts: sharp transitions between hues.
- Loses information in grayscale.





MATLAB Parula Colour Map (2014)



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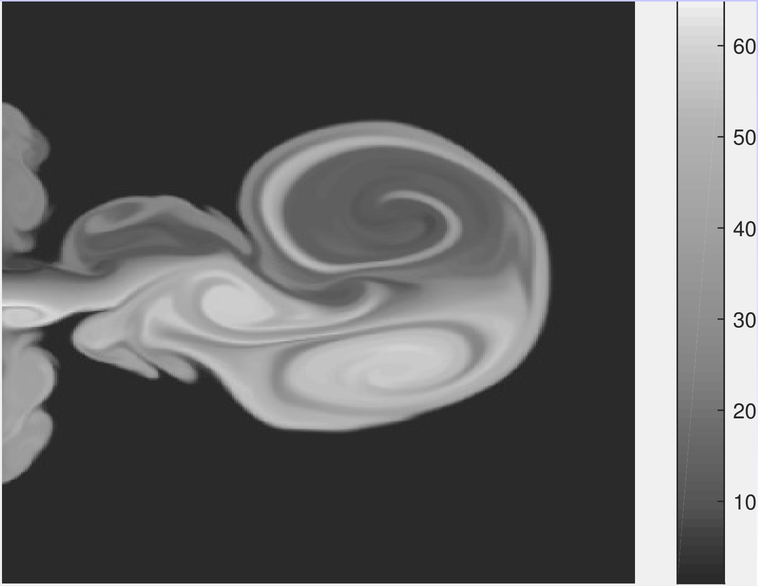
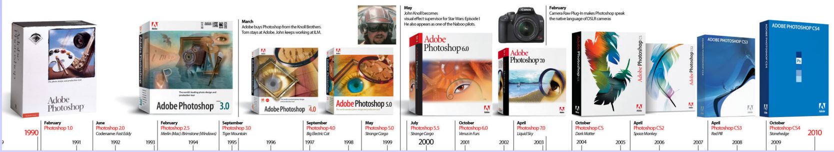


Adobe Photoshop



- Photoshop 1.0 (Mac), 1990.
- Market leader for commercial bitmap/image manipulation.
- Supports RGB, LAB, CMYK.
- Excels in non-destructive editing (layers).
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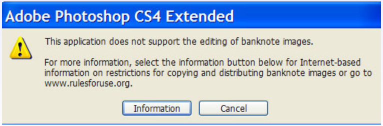

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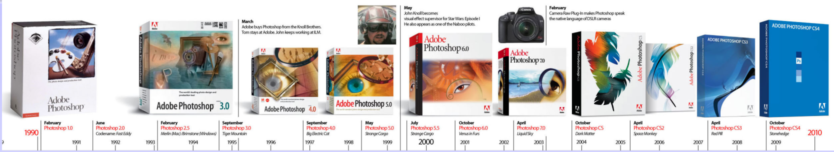

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JPEG

JPEG (1992) stores RGB images in **compressed** form.

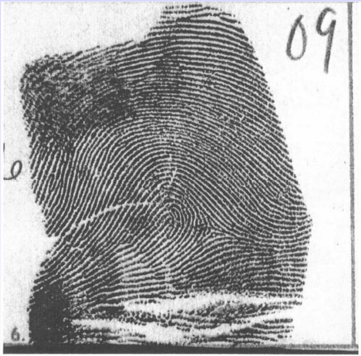
It converts from RGB to YC_bC_r **colour space** where Y = luminance, C_b = blue, C_r = red by

$$\begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ -0.1687 & -0.3313 & 0.5 \\ 0.5 & -0.4187 & -0.0813 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}.$$

- Transformation must be **inverted** to display a JPEG image.
- Human vision more sensitive to **luminance** than **colour**, so can more heavily compress C_b, C_r coordinates.

Fingerprints—FBI

- Digitized at 500dpi $\Rightarrow$ 10Mb. Compression $\gtrsim$ 10:1 req'd.
- Standardized on wavelet compression (1993).
- Jpeg: resonance of 8-pixel tiling w/ 500dpi scans, many edges.
- Wavelets: gradual blurring as compression increased.



Adobe DNG

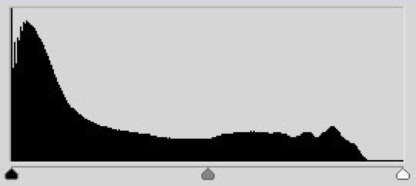
XYZtoCamera matrix is $n \times 3$, $n =$ dim of camera colour space, usually 3 or 4.

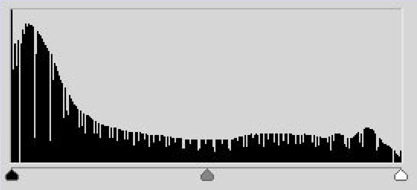
Translating Camera Neutral Coordinates to White Balance xy Coordinates

1. Guess an xy value. Use that guess to find the interpolation weighting factor between the color calibration tags. Find the XYZtoCamera matrix as above.
2. Find a new xy value by computing:
   XYZ = Inverse (XYZtoCamera) * CameraNeutral
   (If the XYZtoCamera matrix is not square, then use the **pseudo inverse**.)
3. Convert the resulting XYZ to a new xy value.
4. **Iterate** until the xy values converge to a solution.

Rounding Errors

- Every editing operation executes $p_{ij} = \text{round}(f_{ij}(p_{ij}))$.
- Rounding errors can potentially cause deterioration.





- Controversy over 8-bit vs. 16-bit editing.
- Controversy over colour space: choice & conversions.

Arithmetic on Images: Brightening

Simple arithmetic on images ($+$,$*$,$-$,$/$) can be very effective!

Let $R, G, B \in [0, 1]$ with

$$\text{black} = (0, 0, 0), \quad \text{white} = (1, 1, 1).$$

To **brighten an image** we need to increase the coordinates.

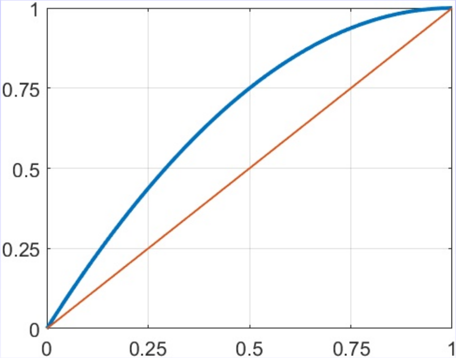
Original

Simple Brightening Transformation

Better Brightening Transformation

Map each coordinate

$$x \leftarrow 1-(1-x)^2.$$



Photoshop: Apply Image with Screen Blending Mode

Pixel-Dependent Brightening

Final Image

Change Autumn into Summer

Looking at the Numbers

Sample colours from photo.

- Typical RGB values for green tree leaves:

$$(R, G, B) = (110, 103, 53),\ (50, 55, 12),\ (135, 125, 81).$$

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- Typical RGB values for yellow tree leaves:

$$(R, G, B) = (250, 193, 73),\ (152, 88, 90),\ (194, 112, 18).$$

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$$(R, G, B) = (110, 103, 53),\ (50, 55, 12),\ (135, 125, 81).$$

- Typical RGB values for yellow tree leaves:

$$(R, G, B) = (250, 193, 73),\ (152, 88, 90),\ (194, 112, 18).$$

Solution

Make $R = G$ by copying the green coordinates into the red.

It's Summer

Original

With Mask to Protect Sky

Repainting University Place

RePainted

Flip Sign of A Channel

- UoM turquoise is $(L, A, B) \approx (85, -12, -3)$.
- Convert to LAB then $A \leftarrow -A$.
- Now have $(L, A, B) \approx (85, 12, -3)$.


KLM
PH-BQD
PH-BDE
PH-BTH
PH-BTB
PH-BDD
Royal Dutch Airlines
The Flying Dutchman
ANTON van DIJK bv


KLM
PH-BDD
Royal Dutch Airlines
The Flying Dutchman
PH-BTB
PH-BTH
PH-BDE
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MANCHESTER
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Car Park B


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Mean

Median

Max

Min

Variance

Summary

- Maths intrinsic to **modelling colour**, and defining, analyzing and exploiting colour spaces.
- Can go a long way in **manipulating the colour of images** with elementary maths.
- All the maths needed to understand colour is covered in the Manchester **honours degree maths programme**.

Talk, including references, available at
`http://www.maths.manchester.ac.uk/~higham/talks/digphot_long.pdf`

Acknowledgements for Graphics

- Wikipedia:
  http://en.wikipedia.org/wiki/Image:CIE1931_XYZCMF.png
  http://upload.wikimedia.org/wikipedia/commons/b/b0/CIExy1931.png
  http://en.wikipedia.org/wiki/Bayer_filter
- http://www2.cmp.uea.ac.uk/Research/compvis/ColourIntro/ColourIntro.htm

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