Euler Characteristic

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1 Introduction

Euler characteristic is a very important topological property which started out as nothing more than a simple formula involving polyhedra. Euler observed that in any polyhedron, the sum of the number of vertices, v, and number of faces, f, was two more than the number of edges, e. In other words, Euler's formula for polyhedra is:

$$v - e + f = 2$$

Since then, the concept of Euler characteristic has transformed into a very useful topological tool which allows mathematicians to study 3-dimensional surfaces as well as surfaces in higher dimensions. In this paper, we will explore the uses of Euler's formula in graph theory, move to higher dimensions and calculate the Euler characteristic of specific surfaces, and also discover how it is useful in homology.

2 Euler Characteristic in Graph Theory

As discussed in [2], the notion of a *graph* was due to Euler in 1736. So let's start with the definition of a graph, as defined in [1].

Definition: A graph, G, consists of two sets: a nonempty finite set V of vertices and a finite set E of edges consisting of unordered pairs of distinct edges from V.

So the basic idea is that we have a set of vertices and we connect vertices of the graph with edges. We say a vertex is *incident* to an edge when there is an edge connecting that vertex to another. Notice there was no specification on the number of edges between vertices or whether every edge had to be incident to an edge. As it turns out, we will only deal with connected graphs (so there are no isolated points) and will only deal with *simple* graphs (those with no more than one edge between any two vertices and no edge connecting a vertex to itself). Before we go into Euler's formula, we must introduce a few more terms.

Definition: A *tree* is a connected graph containing no cycles. That is, it is a connected graph with no set of vertices connected in a closed chain of edges. ([1])

It can be shown that the number of edges in a tree is one less than the number of vertices, and that will be important later in the proof of Euler's formula. Finally, the last idea we will need before we go into the uses of Euler's formula is what it means for a graph to be *planar* (Also defined in [1]).

Definition: A graph is planar if it can be drawn in the plane with no crossing edges. A planar graph which has been drawn is called a *plane graph*.

With these terms defined, we can now move to the statement and proof of Euler's formula (the proof is from [1]).

Theorem 1. *Let G be a connected plane graph with v vertices, e edges, and f faces. Then*

$$v - e + f = 2 \tag{1}$$

An important graph in the proof is called a spanning tree T of G. This is nothing more than a tree which uses every vertex of G but not necessarily all the edges of G. It is a theorem that every connected graph has a spanning tree. I will not prove this here, but a proof can be found in [1]. And with this in mind, we can prove Euler's formula:

Proof. Begin with a spanning tree T of G. For a tree, $e = v - 1$, and $f = 1$ so (1) holds for T. Now G can be obtained from T by adding edges. Each time we add an edge, we also create a new face. So the value of e goes up by 1, but so does the value of f. These contributions cancel in (1), so after every edge is added, equation (1) still holds. □

While you were reading this proof, it may have seemed strange that in the case of T, there was one face. Euler thought of faces as the bounded regions in a plane graph *and* the unbounded region as well, which justifies the $f = 1$ in the tree case.

Right now, all we have is that Euler's formula works for connected plane graphs, not polyhedra. But as it turns out, any polyhedron can be made into a connected plane graph by simply puncturing one of the faces of the polyhedron and flattening it out on the plane. The punctured face becomes the unbounded face. So this shows that Euler's formula holds for polyhedra.

Now, while I don't wish to linger on Euler's formula and would like to introduce the notion of Euler Characteristic, I want to present one application of Euler's formula ([2]). For those who need a quick reminder, a platonic solid is a convex polyhedron where all the faces are congruent regular polygons and the same number of faces meet at each vertex. Note that this means each vertex is incident to the same number of edges as well. Figure 1 shows five platonic solids, and the next theorem asserts that these are the only five.

Theorem 2. *There are exactly five Platonic solids*

Proof. Let n be the number of edges and vertices on each face, and let d be the degree of the vertex (the number of edges each vertex is incident to). Now if we take nf, we are multiplying the number of faces by the number of edges on each face, so we will get twice the number of edges since each edge belongs to two faces. Similarly, if we take dv, since

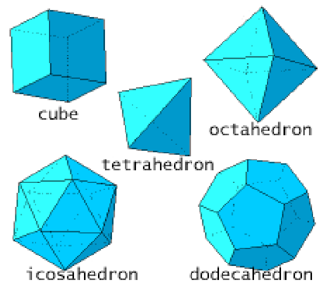

Figure 1: The five platonic solids. Image from http://www.pzzls.com/images/

each edge is incident to two vertices, $dv = 2e$. So $nf = 2e = dv$. Rearranging, we find that $e = \frac{dv}{2}$ and $f = \frac{dv}{n}$ Plugging these into Euler's formula yields $v + \frac{dv}{n} - \frac{dv}{2} = 2$, or

$$v(2n + 2d - nd) = 4n \tag{2}$$

But n and v are positive, so in order for (2) to be satisfied, $2n + 2d - nd > 0$, or $(n-2)(d-2) < 4$. There are only five possibilities for (n, d):
$(3, 3)$ = tetrahedron
$(3, 4)$ = cube
$(4, 3)$ = octahedron
$(3, 5)$ = dodecahedron
$(5, 3)$ = icosahedron

□

3 Higher dimensional surfaces and Euler Characterstic

3.1 Definition and Invariance

To study Euler characteristic in higher dimensions, I will introduce *simplicial complexes* which will allow us to formulate a precise definition of Euler characteristic as well as study its properties.

Definition (from [2]): A simplicial complex, S, is a finite collection of nonempty subsets of a finite set with the property that if $\sigma \in S$ and τ is a nonempty subset of σ, then $\tau \in S$. Any element of S with cardinality $n + 1$ is called an n-simplex.

Note that there is a largest n for which S contains n-simplices, and this is called the dimension of S. S is what is sometimes referred to as an *abstract* simplicial complex. We could define a *geometric* simplicial complex in a similar way. It involves convex hulls, and it is not really necessary to introduce it for this paper, so I will not include it here.

This definition seems strange and can be somewhat confusing. So let's take a brief detour to look at an example of a simplicial complex. Consider the complex:

$$S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$$

Notice that S satisfies the conditions in the definition of a simplicial complex. Now the 0-simplices are what we called vertices earlier. So S has 4 vertices: 1, 2, 3, and 4. The 1-simplices are what were edges before. So there are 6 edges here, {1,2}, {1,3}, {2,3}, {1,4}, {2,4}, {3,4}. 2-simplices were our faces, but now they form filled triangles. So we have 4 faces, namely:

$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$$

Notice that S could have included the 3-simplex {1,2,3,4}, but it didn't. Now if we wanted to picture this, try picturing a hollow tetrahedron. The four vertices are 1,2,3,4. The different edges correspond to our 1-simplices, and the four triangles correspond to our 2-simplices. If we had the 3-simplex {1,2,3,4}, we would have had a solid tetrahedron.

We could apply this to higher dimensional objects. But since those are a bit harder to picture, I won't try and push our imaginations any further. However, with this tool at our disposal, I can finally introduce to you the definition of the Euler Characteristic (as presented in [2]).

Definition Given an n-dimensional simplicial complex S, let c_k (sometimes written $c_k(S)$) be the number of k-simplices in S. The *Euler Characteristic*, χ, is defined as

$$\chi(S) = c_0 - c_1 + c_2 - c_3 + ...$$

You should be able to see that this definition agrees with our earlier idea of Euler Characteristic (although I did not call it that specifically), as c_0 represents the number of vertices, c_1 is the number of edges, and c_2 is the number of faces. There were no 3-simplices so we didn't need c_3.

So we finally defined the Euler characteristic. So what? Do we really have to go triangulating every surface in order to figure out χ? Well, not quite. Luckily, the Euler characteristic is both a topological and homotopy invariant. I present these ideas theorems.

Theorem 3. *The Euler Characteristic is a topological invariant. That is, if X is homeomorphic to Y, then $\chi(X) = \chi(Y)$.*

The proof of this theorem isn't easy, though it becomes a little bit easier after we introduce the idea of homology, but I will not present a proof here.

Theorem 4. *The Euler Characteristic is a homotopy invariant. That is, if X is homotopic to Y, then $\chi(X) = \chi(Y)$.*

I will give a very brief sketch of a proof here, but a full, detailed proof can be found in [6].

Proof. (Sketch) Theorems 5.1 and 5.2 in [6] (pages 197-198) assert that the fundamental group of a connected graph, X, is a free group on n generators, where n is the number of edges not in the maximal tree of X. If we take those two theorems as true, then we can proceed as follows. Notice that if T is a finite tree, then $\chi(T) = 1$ since there is one less edge than vertex (i.e. $c_1 = c_0 - 1$) and no 2-simplices. Now let T be a maximal tree in X, with $e_1, ..., e_k$ the edges in X not in T. Then since $\chi(T) = 1$ and there are k extra edges, $\chi(X) = 1 - k$. On the other hand, $\pi_1(X)$ is a free group on k generators. So $\pi_1(X)$ is a free group on $1 - \chi(X)$ generators. Since the fundamental group is a homotopy invariant, so is $\chi(X)$. □

3.2 Some examples and properties

Now let us see if we can calculate some Euler characteristics and find some more properties as well.

The next calculations are presented in [8]. Let Δ_n be the closed n-dimensional simplex. That is, Δ_0 is a point, Δ_1 is a line segment, Δ_2 is a filled triangle, Δ_3 is a solid tetrahedron, etc. Notice that each of these Δ_n is homotopic to a single point. Since χ is a homotopy invariant and χ of a point is 1,

$$\chi(\Delta_n) = 1$$

Also notice that each of these Δ_n is homeomorphic to D^n (or B^n if you like that notation better), the n-dimensional disk (or ball). Since χ is a homeomorphism invariant,

$$\chi(D^n) = 1$$

Now what about S^n? Well S^0 consists of all points in $\mathbb{R}$ a distance 1 from the origin, so it is two points: $\{\pm 1\}$. So $c_0 = 2$ so $\chi(S^0) = 2$. Now notice that S^n can be constructed by taking two closed hemispheres and joining them at a center $(n-1)$-sphere. So we have

$$\chi(S^n) = 2\chi(D^n) - \chi(S^{n-1}) = 2 - \chi(S^{n-1})$$

Given that $\chi(S^0) = 2$, we can inductively deduce that:

$$\chi(S^n) = \begin{cases} 2, & n \in 2\mathbb{Z} \\ 0, & otherwise \end{cases}$$

Just for the sake of getting comfortable with manipulating with c_i, let's prove that result by a more direct approach from the definition. Notice that S^n is just Δ_{n+1} without the $(n+1)$-simplex. For example, S^1 is homeomorphic to Δ_2 (triangle) without the triangle filled in. So in particular,

$$\chi(S^n) = \sum_{i=0}^{n} (-1)^n c_i(\Delta_{n+1})$$

Notice I did not put in the $(n+1)$ term in the sum. However, we can put it in as long as we subtract it as well, so

$$\chi(S^n) = \sum_{i=0}^{n+1} (-1)^n c_i(\Delta_{n+1}) - (-1)^{n+1} c_{n+1}(\Delta_{n+1})$$

But the first sum on the right side is just $\chi(\Delta_{n+1})$, and the number of $(n+1)$-simplices in Δ_{n+1} is 1, so this sum becomes:

$$\chi(S^n) = \chi(\Delta_{n+1}) + (-1)^n = 1 + (-1)^n$$

From this formula, we can get the same result as before.

Now let's see how to calculate the Euler characteristic of a torus. Figure 2 presents a triangulation of the torus. While making sure to take identifications into account, it should be easy enough to calculate the Euler characteristic of the torus and find that it is 0.

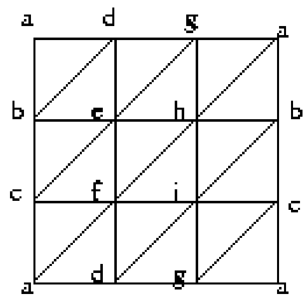

Figure 2: Triangulation of torus. Image from http://blog.mikael.johanssons.org/wp-content/

Now what about connected sums of two surfaces? Well the connected sum removes an open disc from each and then identifies the boundary of the discs to connect the surfaces. In terms of simplicial complexes, this means removing 1 face from each surface, so χ goes down by 1 for each. What happens when we identify the boundary of the triangles? Well we lose 3 vertices, c_0, but we also lose 3 edges, which is c_1, so χ does change when we do this. So the net change in Euler characteristic is -1 for each surface, so -2 for the sum. I present this as a theorem ([3]).

Theorem 5. *For two compact surfaces M_1 and M_2, $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$*

Corollary 6. $\chi(\Sigma_g) = 2 - 2g$, *where Σ_g is the orientable surface of genus g.*

Proof. It was stated earlier that $\chi(T^2) = 0$. Well the orientable surface of genus g is formed by perfoming a connected sum of g torii with the 2-sphere. $\chi(S^2) = 2$. By repeatedly applying theorem 5, we find that adding g torii decreases the Euler characteristic by $2g$, since each of the individual torii contribute nothing. So $\chi(\Sigma_g) = 2 - 2g$. □

Corollary 7. $\chi(\Psi_g) = 2 - g$, *where Ψ_g is the non-orientable surface with g cross caps.*

Proof. It is not hard to verify that $\chi(P^2) = 1$ (triangulate the projective plane and calculate χ directly). Ψ_g is formed by performing the connected sum of P^2 with S^2 g times. So after performing these g connected sums, we find that $\chi(\Psi_g) = 2 + g - 2g = 2 - g$ by theorem 5. □

Observe that if we remove n open discs from either Σ_g or Ψ_g, it is equivalent to removing n faces, so the Euler characteristic just decreases by n.

The next theorem is very nice and it deals with covering spaces (found in [5]).

Theorem 8. *Σ_h is a covering space of Σ_g if and only if $\chi(\Sigma_g) | \chi(\Sigma_h)$*

Proof. We just found that $\chi(\Sigma_g) = 2 - 2g$. Suppose $\chi(\Sigma_g)|\chi(\Sigma_h)$, then for some $n \in \mathbb{N}$, $n(2 - 2g) = 2 - 2h$, so $h = n(g - 1) + 1$. As we proved in one of our homework problems, $\Sigma_{n(g-1)+1}$ is a covering space of $\Sigma_{g-1+1} = \Sigma_g$. For the other direction, suppose Σ_h is an n-sheeted covering of Σ_g. Let T_g be a triangulation of Σ_g. It can be shown that this can be lifted to a triangulation T_h of Σ_h. Each vertex in T_g has n pre-images in Σ_h, and same for edges and faces. So $\chi(\Sigma_h) = n\chi(\Sigma_g)$. □

Before I move on to the final topic of this section, I present the following properties of Euler characteristic without proof ([8]).

Theorem 9. *If X is the union of two closed sets S_1 and S_2, then $\chi(X) = \chi(S_1) + \chi(S_2) - \chi(S_1 \cap S_2)$.*

Theorem 10. *For surfaces X and Y, $\chi(X \times Y) = \chi(X)\chi(Y)$*

3.3 Minimal Triangulations of Surfaces

The last question I want to address before I move on to homology groups deals with triangulations of compact surfaces. In particular, is there a way to find a lower bound on the number of vertices needed in any triangulation of a surface with a known Euler characteristic? The answer is yes. The work I present here can be found in [4].

We know that

$$c_0 - c_1 + c_2 = \chi \tag{3}$$

What can we say about $3c_2$? Well that multiplies the number of faces by the number of edges in each face, but that counts twice the number of edges since every edge belongs to two faces, so $2c_1 = 3c_2$, which means $c_2 = \frac{2c_1}{3}$ Plugging this into (3) gives

$$c_0 - \frac{c_1}{3} = \chi$$

so $c_1 = 3c_0 - 3\chi$. What else do we know about c_1? Well we know the number of edges is at most $\binom{c_0}{2}$ (this bound ensures every pair of vertices has an edge between them). So we have:

$$3c_0 - 3\chi \leq \binom{c_0}{2} = \frac{c_0^2 - c_0}{2}$$

So now we have $c_0^2 - 7c_0 + 6\chi \geq 0$. For this to happen, we must have

$$c_0 \geq \frac{7 + \sqrt{49 - 24\chi}}{2} \tag{4}$$

Thus (4) gives a lower bound on the number of vertices needed in triangulating a surface. For example, for a torus, $\chi = 0$, so (4) says we need at least 7 vertices in the triangulation. And it turns out you can triangulate a torus with 7 vertices and there is no smaller triangulation. This also shows that Figure 2 doesn't give the best triangulation of the torus.

4 Homology Groups and Euler Characteristic

The last topic I want to briefly discuss is that of homology groups. So to do that, I will first introduce the *boundary homomorphism* of a simplicial complex, $\partial_n : C_n(S) \to C_{n-1}(S)$, where C_{-1} is trivial. We already have an intuitive idea of what boundary is. The boundary of the n-simplex should be those $(n-1)$-simplices which together form the "bounds" of the n-simplex. So for example, the boundary of a filled triangle should be the three edges which together form the triangle. This is close to what ∂_n is, but it doesn't take care of orientation, which we will need in these homology groups. So let σ be an n-simplex with vertices $\langle v_0, v_1, ..., v_n \rangle$. Let $< v_0, v_1, ..., \hat{v_j}, ..., v_n >$ be the $(n-1)$ face $< v_0, v_1, ..., v_{j-1}, v_{j+1}, ..., v_n >$ (omit the vertex v_j). We then define the boundary homomorphism ∂_n by:

$$\partial_n(< v_0, ..., v_n >) = \sum_{j=0}^{n} (-1)^j < v_0, v_1, ..., \hat{v_j}, ..., v_n >$$

Also, if the dimension of the simplex is n, we define $\partial_q = 0$ for $q > n$ or $q \leq 0$. Now, it is not hard to show that $\partial_{q-1} \circ \partial_q = 0$ for all q by direct computation, and I will not do it here. However, notice that $\partial_{q-1} \circ \partial_q = 0$ implies that the image of ∂_q is in the kernel of ∂_{q-1}. Because this is true, we can quotient the former by the latter. So we define the q-th homology group, H_q by

$$H_q = Z_q / B_q$$

where $Z_q = \ker\, \partial_q$ and $B_q = \text{im}\, \partial_{q+1}$ ([9]).

Now let β_q be the dimension of H_q. As defined in [7], the β_q are called the betti numbers. Now the main result for this section is the following theorem (from [7]):

Theorem 11. *For an n-dimensional simplicial complex S, $\chi(S) = \sum_{i=0}^{n} (-1)^i \beta_i$*

Proof. The definition of χ is $\sum_{i=0}^{n} (-1)^i c_i$. Since we can regard the space of r-simplices of S as a vector space, the dimension formula holds. So $c_r = \dim(\ker \partial_r) + \dim(\text{im} \partial_r)$, which is $\dim\, Z_r + \dim\, B_{r-1}$, where B_{-1} is defined to be trivial. Also, $\beta_r = \dim Z_r - \dim B_r$ by dimension formula for quotient spaces. From these, we see

$$\chi(S) = \sum_{i=0}^{n} (-1)^i c_i = \sum_{i=0}^{n} \dim Z_r - \dim B_{r-1}$$

Now because B_{r-1} is trivial, we can replace the B_{r-1} with B_r as long as we take into account the sign change, so we have

$$\sum_{i=0}^{n} \dim Z_r - (-1)^r \dim B_r$$

But then this is precisely $\sum_{i=0}^{n} (-1)^i \beta_i$. □

Moreover, since homology groups are a topological invariant, so is Euler characteristic.

It may be worthwhile to work through an example. Let's consider the hollow tetrahedron. It has 4 triangles, $c_2 = 4$, namely

$$\{\{123\},\{124\},\{134\},\{234\}\}$$

First observe that since C_{-1} is trivial (namely, $\{0\}$), every element of C_0 is in the kernel of ∂_0. So we only need to compute ∂_1 and ∂_2. Luckily, we can represent them as matrices. We do this by making all the elements of C_{i-1} into the rows and the elements of C_i the columns. Then the elements of each column representing an element C_i corresponds to the coefficient of the associated C_{i-1} element in the boundary map. So the tetrahedron has 6 edges and 4 faces, so it is a 6×4 matrix. What is the boundary of $\{123\}$? Well we omit each vertex one at a time and take the sign into account as defined in the homomorphism. So it is

$$(-1)^0\{23\} + (-1)^1\{13\} + (-1)^2\{12\}$$

Similarly, the boundary for the face $\{124\}$ is

$$\{24\} - \{14\} + \{12\}$$

We can do the same for the other two faces. So if the columns of the matrix are the 4 faces above in that order and the rows are the edges $\{12\},\{13\},\{14\},\{23\},\{24\},\{34\}$ in that order, then the matrix should look like the following for ∂_2:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Similarly, if we computed the boundary of each of the edges, the matrix of ∂_1 has the edges as the columns and the vertices $\{1, 2, 3, 4\}$ as the rows, and the matrix should look like:

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We can calculate the kernel and image of these matrices by row reduction and by using the dimension formula for vector spaces. So the relevant numbers are: $\beta_0 = \dim(\ker\partial_0) - \dim(\mathrm{im}\partial_1) = 4 - 3 = 1$,
$\beta_1 = \dim(\ker\partial_1) - \dim(\mathrm{im}\partial_2) = 3 - 3 = 0$,
$\beta_2 = \dim(\ker\partial_2)$-$\dim(\mathrm{im}\partial_3) = 1 - 0 = 1$.
So the Euler characteristic is $1 - 0 + 1 = 2$ as we expected.

5 Conclusion

Clearly, the Euler characteristic is a very important topological property because of its invariance. It also has many nice properties that can lead to very nice results, as I hope I was able to show you. There are so many more uses of the Euler characteristic not discussed here. For example, it is very important in the classification of surfaces, as we learned in class. There are also many more ways to use it with simplicial complexes, but that required much more detail than I felt was necessary for this paper. But I hope that at the very least this paper shed some light on the importance and uses of Euler characteristic, as well as the different ways to calculate it.

References

[1] Buckley, Fred, and Marty Lewinter. *A Friendly Introduction to Graph Theory.* Upper Saddle River: Prentice Hall, 2002. Print.

[2] Early, Edward. "On the Euler Characteristic." MIT Undergraduate Journal of Mathematics. 39-40.

[3] Goodman, Sue E. *Beginning Topology.* Pacific Grove: Thomson Brooks, CA. Print.

[4] Hougardy, Stefan, Frank H. Lutz, and Mariano Zelke. "Surface Realization with the Intersection Segment Functional." TS. Universitat Bonn. Web. http://arxiv.org/PS_cache/math/pdf/0608/0608538v2.pdf

[5] Kreek, R. A. *Surface Groups.* TS. Web. http://www.math.uchicago.edu/m̃ay/VIGRE/VIGRE2007/REUPapers/FINALFULL/Kreek2.pdf

[6] Massey, William S. *Algebraic topology, an introduction.* New York: Springer-Verlag, 1977. Print.

[7] Nakahara, Mikio. *Geometry, topology, and physics.* Bristol: Institute of Physics, 2003. Print.

[8] Nicolaescu, Liviu I. "The Euler Characteristic." TS. University of Notre Dame. Web. http://www.nd.edu/l̃nicolae/EulerChar.pdf

[9] Wilkins, David R. *Simplicial Homology Groups.* TS. Trinity College Dublin. Web. http://www.maths.tcd.ie/d̃wilkins/Courses/421/421S6_0809.pdf