# On the Euler Characteristic

EDWARD EARLY

**Abstract.** We discuss the Euler Characteristic and some of the consequences of its topological invariance. We start by following the path of history that motivated the study of this characteristic. Several equivalent definitions are given, along with some advantages of each. We then discuss how the Euler Characteristic is related to graph coloring in the Heawood coloring theorem, continuous tangent vector fields on a manifold in the Poincaré–Hopf index theorem, and differential geometry in the Gauss–Bonnet theorem.

**1. Introduction.** In 1736 Leonhard Euler wrote a paper called "The Seven Bridges of Königsberg," in which he introduced the geometry of position, now known as topology. In particular he introduced the notion of a graph, which will lead to our first definition of the Euler Characteristic. The problem he solved is stated as follows: "In the town of Königsberg (now Kaliningrad) in Prussia there is an island, called 'Kneiphof,' with the two branches of the river (Pregel) flowing around it, as shown in Figure 1-1. There are seven bridges crossing the two branches. The question is whether a person can plan a walk in such a way that he will cross each of these bridges once but not more than once." The impossibility of such a walk had been suspected for some time, but nobody had managed to prove it before Euler.

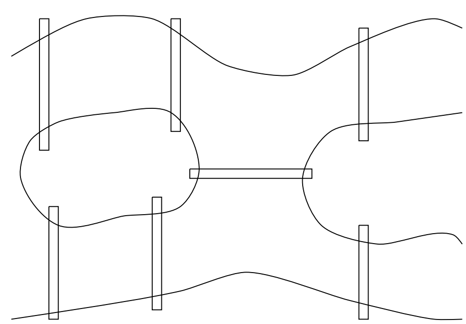

**Figure 1-1.** The seven bridges of Königsberg.

The method Euler used was quite simple in retrospect. He represented the situation as a graph, that is, a collection of points (vertices) and lines (edges) where each edge connects exactly two vertices. In particular, we make the land regions vertices and the bridges edges to get Figure 1-2. He then proved that for a graph to have a walk that uses each edge exactly once there can be at most two vertices with an odd number of edges touching them. The number of edges connected to a vertex is called the *degree* of that vertex. The graph above has four vertices of odd degree, thus our problem is solved.

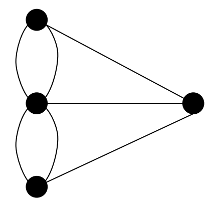

**Figure 1-2.** The seven bridges as a graph.

What Euler noticed was that one could look at geometric problems without any notion of distance. A collection of points and lines was a sufficient model of the bridges to solve the problem. Since graphs will turn out to be useful in other situations, we would like to learn more about them.

The graph above can be drawn in the plane with no edge crossings. Do all graphs have that property? A graph that does is called *planar*, and the answer is that not all graphs are planar. Equivalently, a graph is planar if and only if it can be drawn on a sphere with no edge crossings. One example of a nonplanar graph is the utility graph shown in Figure 1-3. It gets its name from a puzzle invented by Henry Ernest Dudeney, in which three houses and three utility companies are drawn on a sheet of paper and the task is to connect each house to each utility company by lines that never cross (this is a rather frustrating puzzle since the task is impossible). In fact, a theorem of Kuratowski says that any nonplanar graph is basically built from one of two small graphs, namely, the utility graph below, and a pentagon with all of its diagonals. For a more thorough introduction to basic graph theory, see Trudeau [**7**].

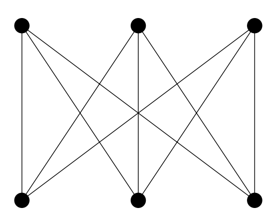

**Figure 1-3.** The utility graph.

In looking at polyhedra, Euler once again turned to graph theory. He noticed that adding the number of vertices and faces of a polyhedron, then subtracting the number of edges, always yielded 2. He mentioned this formula in a letter to Goldbach in 1750, then proved it for convex polyhedra in 1752 by dissecting the solid into tetrahedral slices. This invariance allows one to prove that there are at most five Platonic solids (regular polyhedra), which we will prove as Corollary 2-4.

In Section 2 we will define the Euler Characteristic and explore some of its consequences for planar graphs. We will then extend the definition to higher dimensions in two ways and discuss why each definition is useful. Sections 3 to 5 each contain a discussion of one theorem, whose statement involves the Euler Characteristic. Sections 4 and 5 assume some familiarity with differential geometry of curves and surfaces, for which the reader is referred to Do Carmo [**1**, Ch. 1–4]. The reader is also assumed to be familiar with triangulations, as defined in Munkres [**6**, p. 118], and winding numbers, as defined in Henle [**4**, p. 48].

**2. Definitions.** First we must formalize the ideas in Section 1 so we can give our first definition of the Euler Characteristic.

*Definition* 2-1. A *graph* is a finite set, whose elements are called *vertices*, and a collection of distinct two element subsets, called *edges*.

Geometrically, we can picture a graph as a set of points in $\mathbf{R}^3$ and of continuous curves joining some pairs of them. If the graph is planar, then we can think of the points as being in $\mathbf{R}^2$ or on $S^2$, and choose the curves so that they do not intersect and hence divide the plane or sphere into some number of regions. If the graph is nonplanar, then there exists some orientable surface on which it can be drawn without edge crossings and hence divides that surface into some number of regions. These regions are called *faces*, and we will take it for granted that the number of faces of any graph is well defined. These facts are provable, but we will not demonstrate them.

Let $G$ be a planar graph with $V$ vertices, $E$ edges, and $F$ faces. Since we are requiring that each pair of vertices is connected by at most one edge, Figure 1-2 is no longer considered a graph, though it shares many nice properties with graphs.

*Definition* 2-2. The *Euler Characteristic* $\chi(G)$ is $V - E + F$.

**Theorem 2-3** (Euler's formula). *If $G$ is planar and connected, then $\chi(G) = 2$.*

The proof will not be given here, but can be found in Trudeau [**7**, pp. 97–104].

**Corollary 2-4.** *There are at most five Platonic solids.*

*Proof:* A polyhedron naturally has vertices, edges, and faces, so we can think of it as a graph. This graph is planar since we can draw it on a sphere, and each face is a regular polygon. Let $n$ be the number of edges and vertices on each face, let $d$ be the degree of each vertex, then $nF = 2E = dV$. Rearranging, we get $e = dV/2$ and $f = dV/n$, thus by Euler's formula, we have $V + dV/n - dV/2 = 2$, or $V(2n + 2d - nd) = 4n$. Now, $n$ and $V$ are positive, and hence $2n - 2d - nd > 0$, or $(n-2)(d-2) < 4$. Thus we have only five possibilities for $(d, n)$, namely, (3,3), (3,4), (3,5), (4,3), and (5,3). □

Before moving on to higher dimensions, let us examine another use for this characteristic. In Section 1 we mentioned that not all graphs are planar. Although it is possible to prove a graph isn't planar by using the Jordan Curve Theorem, this can hardly be called an elementary method. With Euler's formula, we can place bounds on $V$, $E$, and $F$ so that any connected graph violating these bounds is nonplanar.

**Theorem 2-5.** *If $G$ is planar and connected with $V \geq 3$, then $\frac{3}{2}F \leq E \leq 3V - 6$.*

*Proof:* If $G$ has a face bounded by fewer than three edges, then the assertion is trivial since, by Euler's formula, the only possibility is $V = 3$, $E = 2$, $F = 1$. Otherwise, we have each face bounded by at least 3 edges. So 3F is less than or equal to the sum of the number of edges bounding each face, which is at most 2E since each edge is counted at most twice. Thus $3F \leq 2E$. Dividing this inequality by 2 yields the first half of the theorem. To get the other inequality, we start with $\frac{3}{2}F \leq E$, add $V - E$ to both sides and apply Euler's formula to the left-hand side to get $2 \leq V - E/3$, or $E \leq 3V - 6$. □

The inequality $3F \leq 2E$ will show up several times in later sections. Note that equality holds if and only if every face is a triangle.

**Corollary 2-6.** *A pentagon with all of its diagonals is nonplanar.*

*Proof:* We have $V = 5$, $E = 10$, so $3V - 6 = 9 < E$. The graph is connected, and hence by Theorem 2-5, cannot be planar. □

If we repeat the argument with the added assumption that no face is a triangle, then we get $2F \leq E \leq 2V - 6$. This inequality is violated by the utility graph, thus showing it is nonplanar. If, instead of looking at the number of edges per face, we look at the number of edges per vertex, then we get another nontrivial result.

**Theorem 2-7.** *If $G$ is planar, then $G$ has a vertex of degree 5 or less.*

*Proof:* If $G$ is not connected, then just look at one of its connected components. If $G$ has fewer than three vertices, then the assertion is trivial. Otherwise, assume each vertex has degree at least 6. Then $6V$ is less than or equal to the sum of the degrees of the vertices, which is exactly $2E$, or $3V \leq E$. Now, by Theorem 2-5, we have $E \leq 3V - 6$, or $3V \leq 3V - 6$, which is absurd, and the result follows. □

The next best thing to a graph in higher dimensions is a simplicial complex. We start with some number of 0-simplices, or points (vertices), connect some pairs of them with 1-simplices, or lines (edges), and so on up to $n$-simplices, where the $(k-1)$-simplices making up the boundary of a new $k$-simplex are called its faces. This concept can be made more precise with a similar definition to that of a graph.

*Definition* 2-8. A *simplicial complex* $S$ is a finite collection of nonempty subsets of a finite set with the property that, if $\sigma \in S$ and $\tau$ is a nonempty subset of $\sigma$, then $\tau \in S$. An element of $S$ of cardinality $n+1$ is called an $n$-simplex.

Since $S$ contains only finitely many simplices, there is a largest $n$ for which $n$ contains $n$-simplices. This value of $n$ is called the *dimension* of $S$. For a more thorough introduction to simplicial complexes see Munkres [**6**, pp. 2–19].

*Definition* 2-9. Given an $n$-dimensional simplicial complex $S$, let $S_k$ be the number of $k$-simplices in $S$. The *Euler Characteristic* is the number,

$$\chi(S) = S_0 - S_1 + S_2 - + \ldots + (-1)^n S_n.$$

It is not hard to see that this definition agrees with our previous definition if we think of a planar graph as a 2-dimensional simplicial complex, so $S_0 = V$, $S_1 = E$, and $S_2 = F$. The nice thing about this definition is that it is simple to compute for a given simplicial complex since all we have to do is count. Thus to find the Euler Characteristic of a surface we need only triangulate it, and think of the vertices, edges, and faces as forming a simplicial complex.

Proving that the Euler Characteristic is a topological invariant, however, is not so easy, so we turn to our second definition. Let $b_k$ be the rank of the $k$th homology group of $S$. A basic fact from homology theory is that $b_k = 0$ if $k$ exceeds the dimension of $S$. For an introduction to homology theory, see Munkres [**6**], Henle [**4**], or Vick [**8**].

*Definition* 2-10. Given an $n$-dimensional simplicial complex $S$, the *Euler Characteristic* is the number,

$$\chi(S) = b_0 - b_1 + b_2 - + \ldots + (-1)^n b_n.$$

The advantage of this definition is that it is not too difficult to prove that the homology groups are topological invariants, thus so is the Euler Characteristic. This definition is equivalent to the first one because of the Hopf trace theorem, which is proved in Munkres [**6**, pp. 122–24].

**3. Graph Coloring.** We will now see how the Euler Characteristic shows up in several theorems. Since we started with graphs, let us first look at the problem of coloring graphs.

*Definition* 3-1. A graph is *colored* if a color has been assigned to each vertex in such a way that no two vertices joined by an edge are given the same color.

The problem we consider is to find the minimum number of colors necessary to color a graph. The four-color theorem asserts that any planar graph can be colored with at most four colors. Although the only known proof of this theorem is so complicated it requires a computer, it turns out the problem is actually easier for nonplanar graphs.

To study nonplanar graphs in any detail, we must first find a surface on which the graph can be drawn without edge crossings. For example, the utility graph mentioned earlier can be drawn on a torus without edge crossings, but not on a sphere. We will always assume a graph has been drawn without edge crossings on some surface. It turns out that it is easier to talk about coloring faces than vertices. Since that is how the problem started (how many colors does a map maker need to give every country a different color from its neighbors?), it seems justifiable to add one more restriction to make the two problems equivalent.

*Definition* 3-2. The *dual* of a graph is obtained by replacing each face with a vertex, and connecting the vertices that correspond to faces that share an edge in the original graph.

From now on, we will assume that all graphs have duals that are also graphs so as to avoid some trivial cases. For example, two vertices joined by an edge, whose dual is one vertex with an edge connecting it to itself, will no longer be considered. With this assumption, coloring the vertices of a graph is equivalent to coloring the faces of its dual graph, so for our proofs we will talk about coloring faces. The main result of this section was obtained by Heawood in 1890.

**Theorem 3-3** (Heawood). *Let $S$ be any surface of characteristic $\chi \leq 0$. Then any graph on $S$ can be colored by $N_\chi$ colors, where*

$$N_\chi = \left\lceil \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rceil .$$

Heawood also conjectured that $N_\chi$ was the minimum number of colors needed to color any graph on $S$ (still assuming $\chi \leq 0$). Alas, in 1934 Franklin proved that only six colors are needed to color every graph on the Klein bottle, not seven as conjectured. After much hard work, the conjecture was settled in 1968 by Ringel and Youngs: it holds in every case except the Klein bottle. It also happens to be true for positive $\chi$, although the case of the sphere, which is equivalent to the four-color theorem, remained unsettled until 1976 when it was proved by Appel and Haken. For a history of the proof, see Harary [**3**, pp. 118–19].

The proof of the Heawood coloring theorem is long, difficult, technical, and will not be reproduced here in its entirety. We will, however, see some of the theorems and lemmas that come into play. For starters, we need to know what kinds of surfaces exist.

**Theorem 3-4** (Classification Theorem). *Every compact, connected surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes.*

This theorem is quite remarkable since it allows us to form a list of all surfaces. There is no known analogous theorem for manifolds in higher dimensions. A proof can be given in six relatively simple steps, and can be found in Massey [**5**, pp. 18–28] or Henle [**4**, pp. 122–27].

Getting back to the theorem at hand, observe that 2E/F is the average number of edges per face of a graph since each edge borders exactly two faces. For proving that $N_\chi$ colors are sufficient, we can use a technical lemma.

**Lemma 3-5.** *Given the positive integer $N$, suppose $2E/F < N$ for all graphs $G$ on $S$. Then $N$ colors are sufficient to color all graphs on $S$.*

*Proof:* The proof is by induction on the number of faces of $G$. If $F < N$, then the assertion is trivial. Suppose it is true for $F = k$, and consider a graph $G$ with $k + 1$ faces. Since the average number of edges per face is strictly less than $N$, there must be a face with fewer than $N$ edges. We can shrink this face to a point by connecting each of its vertices to a point in its interior, then erasing the original edges and vertices of that face. We can now color this $k$-faced graph with $N$ colors by the induction hypothesis, and fewer than $N$ of them are used on the faces adjacent to the one we removed. Thus $G$ can be colored with $N$ colors, and the result follows by induction. □

If a vertex is adjacent to exactly two others, then removing it and making its two neighbors adjacent does not affect the colorability of the graph, nor does it affect the Euler Characteristic. Thus without loss of generality we can assume all vertices have degree at least 3 (in fact, the requirement that the dual be a graph also implies all vertices have degree at least 3), so $V \leq 2E/3$. Replacing $V$ with $\chi - F + E$ and isolating $E$, we get $E \leq 3(F - \chi)$, or $2E/F \leq 6(1 - \chi/F)$. We can apply this inequality immediately to the projective plane, which is the only surface other than the sphere with positive Euler Characteristic.

**Theorem 3-6.** *Any graph on the projective plane can be colored with six or fewer colors.*

The proof that $N_\chi$ colors suffices to color any graph on $S$ now requires just a page of simple algebraic manipulation that will not be reproduced here. The details can be found in Henle [**4**, pp. 172–74]. Proving that we actually need this many colors is significantly harder. The key step, which was not completed until 1968, was to prove that a graph with $V$ vertices, where $V > 2$, that has every pair of vertices adjacent, can be drawn without edge crossings on an $n$-holed torus, where

$$n = \left\lceil \frac{(V-3)(V-4)}{12} \right\rceil .$$

**4. Vector Fields.** One of the most amazing facts about the Euler Characteristic is that it shows up in studies of things that are not related in any obvious way. We now turn our attention to vector fields on a differentiable manifold, as defined in Guillemin and Pollack [**2**].

*Definition* 4-1. A *tangent vector field* $V$ on a manifold $S$ is a continuous function assigning to each point $p$ of $S$ a vector $V(p)$ in the tangent space $T_p(S)$.

To make life easier we will assume that all our tangent vector fields have only *critical points* (that is, points $p$ such that $V(p) = 0$) that are isolated. Not all critical points are the same; Figure 4-1 shows a circulation and a focus, Figure 4-2 shows other things that can happen.

We could ask whether it is possible to have a vector field with a given collection of critical points and no others, such as a sphere with no critical points or just one spiral. Most students of mathematics are probably familiar with the fact that we cannot comb a hairy sphere; in other words, that there is no continuous nonzero tangent vector field on $S^2$, but the fact that it cannot be done with a single circulation is less intuitive.

On the other hand, we can comb a hairy torus rather easily, but for a two-holed torus, intuition doesn't seem to help much. In higher dimensions, visualization simply isn't always possible; so we need something more to see what is going on. To study critical points, we will need some way to tell them apart, and the index is a very effective tool for doing so.

**Figure 4-1.** A circulation and a stable focus.

*Definition* 4-2. Let $M$ be a manifold of dimension $k$, and $V$ a tangent vector field on $M$. For each point $p$ of $M$, the *index* of $V$ at $p$, denoted $I(p)$, is the degree of the map $\frac{V}{|V|} : S_\epsilon \to S^{k-1}$ for $\epsilon$ arbitrarily small.

As luck would have it, the Euler Characteristic tells us what kinds of vector fields we can put on a surface. Note that the index of a noncritical point is just 0; so we will only discuss the index of a critical point.

**Theorem 4-3** (Poincaré–Hopf). *If $V$ is a continuous tangent vector field with isolated singularities on the compact, connected, orientable manifold $M$, then the sum of the indices of the critical points of $V$ is $\chi(M)$.*

Any proof of this theorem requires the development of some machinery to deal with it in higher dimensions (see Vick [**8**] or Guillemin and Pollack [**2**]), but for surfaces we can use more elementary methods. Also, Poincaré only stated the theorem for surfaces. So we will be content to prove only that case here.

By a *surface* we mean a 2-manifold in $\mathbf{R}^3$. In particular, the tangent space at a point is the plane tangent to the surface at that point, and we have the induced metric.

First let us take advantage of the fact that our tangent space is always a plane so we can actually visualize the vector field. We now define the index to be the winding number of the vector field on the boundary of a small neighborhood of the critical point, which agrees with our previous definition. We assume that a small circular neighborhood of a critical point can be divided into finitely many sectors, which can be classified as elliptic, parabolic, or hyperbolic as shown in Figure 4-2. An *elliptic sector* is one in which all paths start and end at the critical point if we follow the vector field. If all the vectors in a sector point toward or away from the critical point, then it is *parabolic*. If there is only one path through the critical point, then the sector is *hyperbolic*.

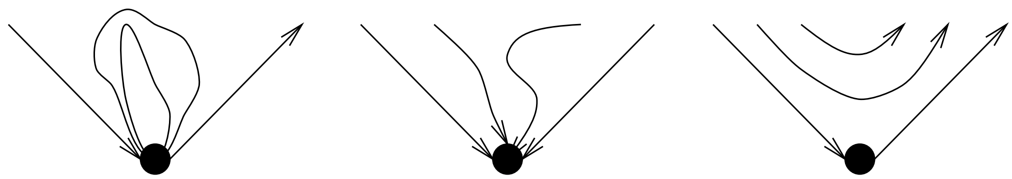

**Figure 4-2.** Examples of elliptic, parabolic, and hyperbolic sectors.

Observe that, as a point $q$ moves through a sector of angle $\theta$, the vector $V(q)$ changes by an angle of $\theta+\pi$ if elliptic, $\theta$ if parabolic, and $\theta-\pi$ if hyperbolic. Thus $I(p) = 1+\frac{e-h}{2}$ where $e$ is the number of elliptic sectors and $h$ is the number of hyperbolic sectors at $p$. With this handy tool and two simple combinatorial lemmas, we will be ready to take on the Poincaré index theorem.

In the following discussion we have a set of vertices, each of which is assigned a label $A$, $B$, or $C$. An edge whose endpoints are labeled $A$ and $B$ is called $AB$. In the case of an oriented surface we have a beginning and ending point of the edge, so $BA$ is not the same as $AB$. If the vertices of a triangle are labeled $A$, $B$, and $C$ in any order, then the triangle is called *complete*.

**Lemma 4-4** (Sperner). *If a triangle with vertices labeled A, B, and C is divided into smaller triangles, and if the vertices of the smaller triangles are each given a label A, B, or C, with the restriction that vertices on AB must be labeled A or B, and similarly for the other two sides, then at least one of the subtriangles receives all three labels.*

*Proof:* In fact, we will see that we get an odd number of complete triangles. First consider the side $AB$ of the original triangle. Let $a$ be the number of segments $AA$, let $b$ be the number of edges $AB$, and let $c$ be the number of vertices labeled $A$ on the interior of the side. Counting yields $2a + b = 2c + 1$, thus $b$ is odd. Now, we look at subtriangles of the original triangle. Let $d$ be the number of triangles $ABA$ or $BAB$, let $e$ be the number of triangles $ABC$, and let $f$ be the number of edges $AB$ (excluding those on the boundary). Counting yields $2d + e = 2f + b$, thus $e$ is odd and hence positive, as desired. □

**Lemma 4-5.** *Let $S'$ be an oriented surface with boundary. If $S'$ has a triangulation with vertices labeled A, B, or C, then the number of complete triangles counted with orientation equals the number of edges AB on the boundary counted with orientation.*

*Proof:* Count a $+1$ for each positively oriented edge $AB$, and $-1$ for each negatively oriented one, and similarly for the complete triangles. Then the sum of the edges $AB$ on the boundary is equal to the sum of all edges $AB$ since those in the interior are counted both $+1$ and $-1$. The latter sum is equal to the sum of the complete triangles since every triangle $ABA$ or $BAB$ contributes $+1$ and $-1$ to the count of edges $AB$, and other triangles do not contain an edge $AB$. □

**Theorem 4-6** (Poincaré). *If $V$ is a continuous tangent vector field with isolated singularities on the compact, connected, orientable surface $S$, then the sum of the indices of the critical points of $V$ is $\chi(S)$.*

*Proof:* The idea is to construct another vector field for which we can easily calculate the indices, then show that the sum of its indices is the same as for $V$. With this idea in mind we turn our attention to a very special type of vector field, namely, the gradient of the function that assigns to each point of $S$ the distance from that point to a fixed plane not intersecting $S$. Note that this vector field gives the direction in which we would run on the surface to get away from the plane as quickly as possible. In this case we have no elliptic sectors, since running away from a fixed object never brings us back to the same point, so $I(p) = 1 - h/2$. Consider a small circle on $S$ around a critical point $p$. It intersects the tangent plane exactly $h$ times. Although this result is not trivial, it is believable enough that we will just assume it.

We choose our fixed plane to give us a gradient field $U$ such that no critical point of $V$ is a critical point of $U$, which we can do since $S$ is compact and the critical points are isolated; hence, there are only finitely many. We now triangulate $S$, including the critical points of $U$ among the vertices, in such a way that no triangle has two vertices the same distance from the fixed plane. Since we have only finitely many vertices to worry about, this triangulation can always be accomplished, perhaps by distorting the surface slightly, without changing the Euler Characteristic. The latter condition makes one vertex of each triangle the middle vertex. So if a point $q$ has $h$ hyperbolic sectors (a noncritical point has $h = 2$), then for a sufficiently fine triangulation $h$ is also equal to the number of triangles with $q$ as the middle vertex. Let $X$ be the set of critical points of $U$. Then $\sum_{q \in X} I(q)$ is equal to the sum over $X$ of 1 minus half the number of triangles with that vertex as the middle. The latter sum is equal to the sum over all vertices of the triangulation of the same quantity, but that's just $V - F/2$. Since we have a triangulation, $3F = 2E$, or $F = 2E - 2F$. Thus the sum of the indices of $U$ is $V - E + F = \chi(S)$, as desired.

We remove a small neighborhood of each critical point of $U$ or $V$ to obtain a new surface $S'$, on which $U$ and $V$ are nonzero. Triangulate $S'$, and for each vertex $p$ of this triangulation, assign a label $A$, $B$, or $C$ depending on how $V(p)$ relates to $U(p)$, as shown in Figure 4-3 with $U(p)$ in the direction of the positive $y$ axis. The positive and negative $x$ axis are part of the $C$ region, the positive $y$ axis is part of the $A$ region, and we need not worry about the origin since we are on $S'$. Note that the orientability of $S$ allows us to do this labeling consistently.

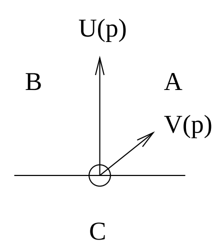


**Figure 4-3.** How to label: $p$ is assigned $A$.

If we had any complete triangles, then we could subdivide to get smaller complete triangles by Sperner's lemma. The intersection of such a sequence of nested closed sets is nonempty since $S'$ is compact, so by the continuity of $V$ we would get a point $p$ in $S'$ for which $V(p) = 0$, contradicting the construction of $S'$. Thus we have no complete triangles, so the sum of the oriented edges $AB$ on the boundary of $S'$ is 0 by Lemma 4-5.

Remember we cut out a small neighborhood of each critical point, and the boundary of that neighborhood is part of the boundary of $S'$. For a sufficiently fine triangulation, the sum of the edges $AB$ around a critical point $p$ of $V$ is the index of $p$, since we are just calculating the winding number (since $U$ remains essentially constant in this neighborhood).

If we had done our labeling by switching the roles of $U$ and $V$, then looking at Figure 4-3 we see without difficulty that the labels $A$ and $B$ are switched, but $C$ is left unchanged. Thus the sum of the edges $AB$ around a critical point $q$ of $U$ is the negative of the index of $q$. In other words, the sum of the indices of the critical points of $V$ minus the sum of the indices of the critical points of $U$ is the sum of the edges $AB$ on the boundary of $S'$, which is 0. Thus the sum of the indices of the critical points of $V$ is equal to the sum of the indices of the critical points of $U$, which is $\chi(S)$. □

The condition of orientability is necessary for this proof to make any sense. Fortunately we can use covering spaces to see that all is not lost if we look at a sufficiently nice nonorientable surface, namely, the projective plane.

**Corollary 4-7.** *Let $V$ be a continuous tangent vector field with isolated singularities on the projective plane. Then the sum of the indices of the critical points of $V$ is* 1*.*

*Proof:* The sphere $S^2$ is the universal covering space of the projective plane, and is a two-sheeted covering. Thus, given a vector field on the projective plane, we can use two copies of it to put a vector field on $S^2$. Now, the sum of the indices of the vector field on $S^2$ is 2 by the Poincaré index theorem. Hence, by the Poincaré index theorem, the sum of the indices of $V$ must be 1, which is the Euler Characteristic of the projective plane. □

**5. Curvature.** We now turn our attention to one more, apparently unrelated, place where the Euler Characteristic appears, namely, the study of curvature of an orientable surface $S$ in $\mathbf{R}^2$. The *Gauss map* of $S$ is the map from $S$ to the sphere $S^2$ that sends a point $p$ of $S$ to the unit normal vector of $S$ at $p$. The *Gaussian curvature* $K(p)$ of $S$ at $p$ can be defined as the limit as $\epsilon$ approaches 0 of the ratio of the area of the image under the Gauss map of an $\epsilon$-neighborhood of $p$, counted positive or negative depending on whether the Gauss map preserves or reverses orientation, to the area of the original neighborhood. Alternatively, we can define $K(p)$ to be the product of the minimum and maximum values of the second fundamental form on the unit circle in $T_p(S)$. If $\gamma$ is a path on $S$, then its curvature vector $\gamma''$ at a point $p$ has a component in the direction of $N \times \gamma'$, where $N$ is the normal vector to $S$ at $p$. The magnitude of this component is called the *geodesic curvature*, denoted $k_g(p)$, of $\gamma$.

A *triangle* on $S$ consists of three paths, $\gamma_1$, $\gamma_2$, and $\gamma_3$, each of which shares each endpoint with one of the other two. Define $\theta_1$ to be the angle between $\gamma_1'$ and $\gamma_2'$ at their common point (note this angle is well defined since these vectors lie in the same tangent plane) and similarly define $\theta_2$ and $\theta_3$.

**Theorem 5-1** (Local Gauss–Bonnet). *Let $S$ be a regular, oriented surface and $T$ a triangle on $S$ whose edges are parametrized by arc length and agree with the orientation of $S$. Then the integral of $k_g$ along the boundary of $T$ plus the sum of $\theta_1$, $\theta_2$, and $\theta_3$, plus the integral over the interior of $T$ of the Gaussian curvature of $S$ is $2\pi$.*

For a proof of this theorem, see do Carmo [**1**, pp. 269–70].

If we triangulate $S$ and apply the local Gauss–Bonnet theorem to each triangle, then the integrals of the geodesic curvature will cancel when added together since they are counted once in each direction. Each $\theta_i$ is an external angle of a triangle, corresponding to the interior angle $\psi_i$, where $\psi_i = \pi - \theta_i$, and each vertex is surrounded by $\psi_i$'s that add up to $2\pi$. So replacing $\theta_1$ with $\pi - \psi_i$, moving everything but the integral of $K$ to the right-hand side, and summing over all triangles, we get that the integral over $S$ of $K$ is $-\pi F + 2\pi V$. Next, using the fact that we have a triangulation, so $2E = 3F$, or $F/2 = E - F$, we can rearrange this expression to include the Euler Characteristic once again. Thus we get our final result.

**Theorem 5-2** (Global Gauss–Bonnet). *Let $S$ be an oriented, closed, compact, connected, regular surface. Then the global integral of the Gaussian curvature of $S$ is $2\pi\chi(S)$.*

## References

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