A (Mostly) Linear Algebraic Introduction to Quaternions

Joe McMahon
Program in Applied Mathematics
University of Arizona

Fall 2003

1 Some History

1.1 Hamilton's Discovery and Subsequent Vandalism

Having seen that complex numbers of unit modulus rotate the complex plane via multiplication, Irish mathematician William Rowan Hamilton sought an analogous structure to provide rotaions of $\mathbb{R}^3$. His introduction of a second imaginary unit, to produce numbers of the form $x_0 + x_1 i + x_2 j$, where $i^2 = j^2 = -1$, was not enough.

After more than ten years of consideration, Hamilton had an epiphany on October 16th, 1843, as he and his wife were walking past Broome Bridge in Dublin. He realized that a system with three imaginary units would suffice, and he carved

$$\begin{aligned} i^2 = j^2 &= k^2 = -1 \\ ij = k & \quad ji = -k \end{aligned}$$

into the stone of the bridge. He created the label quaternion for numbers of the form $q = ai + bj + ck$, where a, b, c, and d are real. I'll show later on how the quaternions provide rotations of $\mathbb{R}^3$.

1.2 Quaternions versus Vectors

Note that addition of quaternions is as simple as addition of complex numbers, but that multiplication is not commutative, thanks to the properties of the imaginary units i, j, k:

$$\begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j \end{aligned}$$

Compare these relations to the cross products $\mathbf{i}\times\mathbf{j}$, $\mathbf{j}\times\mathbf{k}$, and $\mathbf{k}\times\mathbf{i}$ of elementary vector calculus.

Consider the product of two quaternions $u = u_0 + u_1 i + u_2 j + u_3 k$ and $v = v_0 + v_1 i + v_2 j + v_3 k$:

$$\begin{aligned} uv &= (u_0 + u_1 i + u_2 j + u_3 k)(v_0 + v_1 i + v_2 j + v_3 k) \\ &= u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3 \\ &\quad (u_0 v_1 + u_1 v_0 + u_2 v_3 - u_3 v_2) i \\ &\quad (u_0 v_2 + u_2 v_0 + u_3 v_1 - u_1 v_3) j \\ &\quad (u_0 v_3 + u_3 v_0 + u_1 v_2 - u_2 v_3) k \end{aligned}$$

Hamilton's first interest was in using quaternions to model rotations of $\mathbf{R}^3$. He labeled quaternions with zero real part ($u = u_1 i + u_2 j + u_3 k$) pure quaternions or vectors. The product of two vectors is found by setting $u_0 = v_0 = 0$ in the previous result:

$$\begin{aligned} uv &= (u_1 i + u_2 j + u_3 k)(v_1 i + v_2 j + v_3 k) \\ &= (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_3) k - (u_1 v_1 + u_2 v_2 + u_3 v_3) \end{aligned}$$

Compare this to the dot product and cross product of two elements of $\mathbf{R}^3$:

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \end{aligned}$$

The algebra of quaternions indirectly includes all of the algebra of elements of $\mathbb{R}^3$, and more. But quaternions fell out of favor after Josiah Willard Gibbs of Yale University introduced the dot product and cross product in 1881, in his text *The Elements of Vector Analysis*. Hamilton, who had died in 1865, had labored for years on extending the theory of quaternions and the rotations they represent, but he had constructed a difficult noitational structure and had made some fundamental errors in describing which rotation a quaternions represents. Hamilton staunchly defended his anlysis and alienated some in the process.

Nevertheless, some stood by quaternions, and for some the preference of one system versus the other became a partisan split:

> "Even Prof. Willard Gibbs must be ranked as one the retarders of quaternions progress, in virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster, compounded of the notation of Hamilton and Grassman."
> —Peter Guthrie Tait (one of Hamilton's former students), 1890

> "Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell."
> —William Thomson, Baron Kelvin of Largs, 1892

The International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics published a bulletin from 1900 to 1923, but Gibbs's notation by and large replaced quaternions. The overwhelming majority of texts of introductory vector algebra or vector calculus make no mention of the quaternions. They are mentioned in Herbert Goldstein's classic *Classical Mechanics* (on the reading list of every graduate student of physics), but only in a dismissive footnote in the lesson on representing rotation of $\mathbb{R}^3$ with Pauli spin matrices:

> The connossieur of somewhat musty mathematics will recognize in Eq. (4-74) a representation of $\mathbf{Q}$ as a matrix *quaternion*, a quantity invented by Sir William R. Hamilton in 1843.

1.3 Some Prehistory

Although these quantities are most often called Hamilton's quaternions, Hamilton was not the first to discover or even the first to publish results related to them. A look at Gauss's notes reveals that Gauss discovered the quaternions in 1819 but never bothered to publish.

In his doctoral thesis, Benjamin Olinde Rodrigues produced an integral formula, now presented as Rodrigues's formula in many texts of complex analysis, for the Legendre polynomials. Although he has left mathematics for banking long before 1840, in that year he pubslihed a paper that addressed vectorial representations of rotations of $\mathbb{R}^3$. He employed spherical trigonometry to produce a formula for the axis and angle of the single rotatcreatedproduced a parametrization of the quaternions. Though his results preceded those of Hamilton, he was long ignored. But today Rodrigues gets much credit in the history of quaternions.

2 Some Algebra of the Quaternions

The quaternions have addition, defined just as in the complex numbers. Quaternion addition satisfies all the usual requirements of associativity and commutativity. The quaternion 0 is the additive identity element, and each quaternion has an additive inverse. One can also show that the quaternions form a linear (vector) space over the reals.

The quaternion 1 is the multiplicative identity element and is distinct from the additive identity element 0. We will use the notation $||q||$ for the norm of a quaternion q: $||q|| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}$. With this, we can find a formula for the multplicative inverse of any nonzero quaternion q. Let $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$. Then

$$\begin{aligned}
q\frac{\bar{q}}{||q||^2} &= (q_0 + q_1 i + q_2 j + q_3 k) \cdot \frac{(q_0 - q_1 i - q_1 j - q_2 k)}{||q||^2} \\
&= \frac{q_0^2 - q_0(q_1 i + q_2 j + q_3 k) + q_0(q_1 i + q_2 j + q_3 k) - q_1^2 i^2 - q_2^2 j^2 - q_3^2 k^2}{||q||^2} \\
&= \frac{q_0^2 + q_1^2 + q_2^2 + q_3^2}{||q||^2} \\
&= 1
\end{aligned}$$

Hence, every nonzero quaternion has a multiplicative inverse. Scalar mutliplication is distributive and commutes with quaternion multiplication, just as for the complex numbers. The one aspect of the complex numbers that the quaternions lack is commutativity of multiplication. The quaternions satisfy every requirement of a field except that. Such structures are called division rings, division algebras, or skew fields.

3 Quaternions as Rotations

Note that Hamilton's algebra for the quaternions treats each quaternion as both an operator and an operand. Left multiplication by a quaternion q is a linear map $u \mapsto qu$ from the quaternions into the quaternions, as is right multiplication, $u \mapsto uq$. Since these multiplications are linear maps from four-dimensional vector space into itself, we can find a matrix representation of each. We will avoid some headaches in the following analysis if we do so.

Let the quaternion u be represented by the real vector $(u_0, u_1, u_2, u_3)^T$. Note that the norm of the quaternion is easily related to this dot product of the real vector with itself:

$$||u||^2 = u_0^2 + u_1^2 + u_2^2 + u_3^2 = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Since left- and right-muliplication are linear transformations on a four-dimensional vector space, we should be able to represent the products qu and uq by products $L_q u$ and $R_q u$, where L_q and R_q are real 4×4 matrices. The n^{th} column of L should represent the image of the n^{th} unit multiplied on the left by q. For example,

$$qi = q_0 i + q_1 ii + q_2 ji + q_3 ki = -q_1 + q_0 i + q_3 k - q_2 k,$$

and since i is the second unit, the second column of L_q should be

$$\begin{pmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{pmatrix}.$$

Further steps will reveal that the appropriate matrices are

$$\begin{aligned} L_q &= \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \\ R_q &= \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \end{aligned}$$

Note that L_q and R_q are not transposes of each other. One is converted to the other by transposition of the lower right 3×3 submatrix. Note also that

$$\begin{aligned} L_q^T L_q &= \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \\ &= \begin{pmatrix} q_0^2+q_1^2+q_2^2+q_3^2 & 0 & 0 & \\ 0 & q_0^2+q_1^2+q_2^2+q_3^2 & 0 & 0 \\ 0 & 0 & q_0^2+q_1^2+q_2^2+q_3^2 & 0 \\ 0 & 0 & 0 & q_0^2+q_1^2+q_2^2+q_3^2 \end{pmatrix} \\ &= ||q||^2 I \end{aligned}$$

$$
\begin{aligned}
R_q^T R_q &= \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \\
&= \begin{pmatrix} q_0^2+q_1^2+q_2^2+q_3^2 & 0 & 0 & \\ 0 & q_0^2+q_1^2+q_2^2+q_3^2 & 0 & 0 \\ 0 & 0 & q_0^2+q_1^2+q_2^2+q_3^2 & 0 \\ 0 & 0 & 0 & q_0^2+q_1^2+q_2^2+q_3^2 \end{pmatrix} \\
&= ||q||^2 I
\end{aligned}
$$

For a quaternion u, let $[u]$ be the corresponding element of $\mathbb{R}^4$. Then $||u||^2 = [u] \cdot [u] = [u]^T[u]$, where $[u]^T$ is the transpose of $[u]$. We will now show that the norm of a product is equal to the product of norms. Let $v = qu$. Then $[v] = L_q[u]$, and

$$
\begin{aligned}
||qu||^2 &= ||v||^2 \\
&= [v]^T[v] \\
&= (L_q[u])^T L_q[u] \\
&= [u]^T L_q^T L_q[u] \\
&= [u]^T ||q||^2 I[u] \\
&= ||q||^2 [u]^T[u] \\
&= ||q||^2 ||u||^2
\end{aligned}
$$

The same result holds for uq.

We can show by iteration that $||qup|| = ||q|| \cdot ||u|| \cdot ||p||$. Hence, for any quaternions q, u,

$$||quq^{-1}|| = ||q|| \cdot ||u|| \cdot ||q^{-1}|| = ||u|| \cdot ||qq^{-1}|| = ||u|| \cdot ||1|| = ||u||$$

In the language of geometry, the map $u \mapsto puq^{-1}$ is a congruence of Euclidean four-space.

What effect does such a transformation have on the pure quaternions, which we treat as $\mathbb{R}^3$? First note that such a map fixes 1: $1 \mapsto q1q^1 = 1$. If such a map also fixes the pure quaternions, then it is also a congruence of Euclidean three-space. We will show that such a map sends the orthogonal complement of $span 1$ (a slight abuse of notation, as the dot product is not really defined on the quaternions themselves) back into itself. That is, if u is a pure quaternion, then so is quq^{-1}.

Suppose u is a pure quaternion. Then $[u]$ is orthogonal to $[1]$: $[u]^T[1] = 0$. Now consider the quaternion quq^{-1}, whose representative in $\mathbb{R}^4$ is $[quq^{-1}] = L_q R_{q^{-1}}[u]$. Note also that since 1 is fixed by such maps, $[1] = L_q R_{q^{-1}}[1]$.

$$\begin{aligned}
[quq^{-1}]^T[1] &= (L_q R_{q^{-1}}[u])^T L_q R_{q^{-1}}[1] \\
&= [u]^T R_{q^{-1}}^T L_q^T L_q R_{q^{-1}}[1] \\
&= [u]^T R_{q^{-1}}^T ||q||^1 I R_{q^{-1}}[1] \\
&= ||q||^2 [u]^T R_{q^{-1}}^T R_{q^{-1}}[1] \\
&= ||q||^2 [u]^T ||q^{-1}||^2 I[1] \\
&= ||q||^2 ||q^{-1}||^2 [u]^T[1] \\
&= [u]^T[1] \\
&= 0
\end{aligned}$$

So far we have seen that $u \mapsto quq^{-1}$ preserves norm and maps vectors (pure quaternions) to vectors. We can show that it preserves the angle between two vectors. Recall that the vector θ between elements $\mathbf{u}, \mathbf{v}$ of $\mathbb{R}^3$ satisfies

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$$

Given nonzero vectors u, v, with representations $[u], [v]$, let θ be the angle between them, *i.e.*

$$\cos\theta = \frac{[u]^T[v]}{||[u]|| \cdot ||[v]||}$$

Then the cosine of the angle between quq^{-1} and qvq^{-1} is

$$\begin{aligned}
\frac{(L_q R_{q^{-1}}[u])^T L_q R_{q^{-1}}[v]}{||L_q R_{q^{-1}}[u]|| \cdot ||L_q R_{q^{-1}}[v]||} &= \frac{[u]^T R_{q^{-1}}^T L_q^T L_q R_{q^{-1}}[v]}{||[u]|| \cdot ||[v]||} \\
&= \frac{||q||^2 ||q^{-1}||^2 [u]^T[v]}{||[u]|| \cdot ||[v]||} \\
&= \frac{[u]^T[v]}{||[u]|| \cdot ||[v]||} \\
&= \cos\theta
\end{aligned}$$

The angle between any two vectors is preserved.

If q is a vector, then we can easily show that the map $u \mapsto quq^{-1}$ fixes some vector: $qqq^{-1}q$ (it also fixes any real multiple of q). But the map fixes some vector whether or not q is itself a vector. Before we prove this, note that

$$quq^{-1} = \frac{qu\bar{q}}{||q||^2} = \frac{q}{||q||} u \frac{q^{-1}}{||q||},$$

where $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$ is the quaternion conjugate and has the same norm as q. We see from this that the map $u \mapsto quq^{-1}$ is also achieved by the normalized vector $q/||q||$. Henceforth any quaternions used to form such maps are assumed to have unit norm. This will simplify the work significantly. With this assumption, we have $q^{-1} = \bar{q} = q_0 - q_1 i - q_2 j - q_3 k$.

If q is a unit quaternion, then $q_0^2 + q_1^2 + q_2^3 + q_3^2 = 1$, so $q_0 = \cos^2\theta$ and $q_1^2 + q_2^2 + q_3^3 = \sin^2\theta$ for some real θ. Hence,

$$v = \frac{q_1 i + q_2 j + q_3 k}{|\sin\theta|}$$

is a pure quaternion with unit norm. Since $|q_0| = |\sin\theta|$, $q_0 = \pm\cos\theta$. If $q_0 = -\cos\theta$, then $q_0 = \cos(\theta+\pi) = \cos\theta'$, and $q = \cos\theta' + \sin\theta' v$. Such an expression can be found for any unit quaternion q. The expresion for q^{-1} is $\cos\theta - \sin\theta v$, as $q^{-1} = q_0 - q_1 i - q_1 j - q_3 k$ for unit quaternions.

We will use this above expresion to show that each map $u \mapsto quq^{-1}$ fixes a pure quaternion. Suppose $q = \cos\theta + \sin\theta v$. The unit vector v is mapped to

$$\begin{aligned}(\cos\theta + \sin\theta v)v(\cos\theta - \sin\theta v) &= \cos^2\theta v + \sin\theta\cos\theta v - \cos\theta\sin\theta v - \sin^2\theta vvv \\ &= \cos^2\theta - \sin^2\theta vvv\end{aligned}$$

We can find vv by computing $L_v[v]$:

$$\begin{aligned} L_v[v] &= \begin{pmatrix} 0 & -v_1 & -v_2 & -v_3 \\ v_1 & 0 & -v_3 & v_2 \\ v_2 & v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} -v_1^2 - v_2^2 - v_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{1}$$

which corresponds to the quaternion -1. Thus, $vvv = -v$

Hence, the image of v is

$$\cos^2\theta v - \sin^2\theta(-v) = \cos^2\theta v + \sin^2\theta v = v$$

This result has concrete geometric siginificance. It tells us that the map $u \mapsto quq^{-1}$ is a rotation of $\mathbb{R}^3$, but it also tells us that **the axis of rotation lies along the pure vector** v. Now, given any quaternion, we can immediately determine the axis of rotation of the corresponding rotation.

What is the angle of rotation? Since v is fixed by the rotation, it is the plane perpendicular to v (another abuse of notation) that is rotated, as angles (including right angles) are preserved by the map. Let's look at a specific, convenient example. Suppose that $v = i$, and that p is a unit vector in the plane orthogonal to i (another slight abuse of notation). Then $p = \cos\alpha j + \sin\alpha k$.

$$
\begin{aligned}
qpq^{-1} &= (\cos\theta + \sin\theta i)(\cos\alpha j + \sin\alpha k)(\cos\theta - \sin\theta i) \\
&= (\cos\theta\cos\alpha j + \cos\theta\sin\alpha k + \sin\theta\cos\alpha ij + \sin\theta\sin\alpha ik)(\cos\theta - \sin\theta i) \\
&= (\cos\theta\cos\alpha j + \cos\theta\sin\alpha k + \sin\theta\cos\alpha k - \sin\theta\sin\alpha j)(\cos\theta - \sin\theta i) \\
&= (\cos(\theta + \alpha)j + \sin(\alpha + \theta)k)(\cos\theta - \sin\theta i) \\
&= \cos(\theta + \theta)\sin\theta ki \\
&= \cos(\alpha + \theta)\cos\theta j - \cos(\alpha + \theta)\sin\theta k + \sin(\alpha + \theta)\cos\theta k - \sin(\alpha + \theta)\sin\theta j \\
&= \cos(\alpha + 2\theta)j + \sin(\alpha + 2\theta)k
\end{aligned}
$$

The map rotates the orthogonal plane by an angle of 2θ. We can show that this holds in general. We can map i to general vector $u = qiq^{-1}$ by some quaternion q; under this map, every vector p orthogonal to i is mapped to a vector w orthogonal to u. Then the rotation corresponding to $u = qiq^{-1}$ is

$$
\begin{aligned}
w = qpq^{-1} &\mapsto uwu^{-1} \\
&= qiq^{-2}qpq^{-1}(qiq^{-1})^{-2} \\
&= qipq^{-1}qi^{-1}q^{-1} \\
&= qipi^{-1}q^{-1} \\
&= q(\cos(\alpha + 2\theta)j + \sin(\alpha + 2\theta)k)q^{-1}
\end{aligned}
$$

Since these maps preserve angles, the angle between w and qwq^{-1} is equal to the angle between the respective pre-images p and ipi^{-1}: 2θ.

Now, given a quaternion q, we can determine the axis of rotation and the rotation angle caused by the map $u \mapsto quq^{-1}$ on the vectors. Better yet, since we have a simple way to multiply quaternions, **if we have two rotations achieved by quaternion conjugation, we can quickly determine the** *single* **rotation equivalent to a composition of the rotations. This result removes much of the complexity of analysizing the composition of rotations. Programmers working in computer graphics have made great use of these results**.

That the map rotates by 2θ may come as a surprise. Hamilton himself appears not to have made this connection. He viewed quaternion multiplication ($u \mapsto qu$), not quaternion conjugation ($u \mapsto quq^{-1}$), as a rotation operation. Hamilton believed his work and its relation to geometry to have cosmic significance, and he held fast to his interpretations. His refusal to budge from this view contributed to quaternions' fall from favor. One may find it interesting that Rodrigues, forgotten or ignored for a long time, understood the relation between the quaternion angle θ and the angle of the corresponding rotation.

4 Quaternions as Examples in Advanced Mathematics

4.1 Unit Quaternions Form a Lie Group

The quaternions can be used to classify wide variety of discrete symmetry groups of $\mathbb{R}^3$. But showing that the unit quaternions form a very special group is easier. Note that the unit quaternions form a group (albeit not an abelian one) under multiplication, and that when viewed as the unit sphere in $\mathbb{R}^4$, they form a smooth parametrized surface. We can show without much effort that multiplication of the unit quaternions is smooth, so the unit quaternions form a Lie group. The only unit spheres that are Lie Groups are S^1 (the unit complex numbers), S^3 (the unit quaternions), and S^7. We will see later on how inimately these three are related, via algebraic and geometric properties.

4.2 Vectors (Pure Quaternions) Form a Lie Algebra

Consider the commutator $[q,p] = qp - pq$ of two vectors (pure quaternions) q, p.

$$\begin{aligned}
qp &= (q_2p_3 - q_3p_2)i + (q_3p_1 - q_1p_3)j + (q_1p_2 - q_2p_1)k - q_1p_1 - q_2p_2 - q_3p_3 \\
pq &= (p_2q_3 - p_3q_2)i + (p_3q_1 - p_1q_3)j + (p_1q_2 - p_2q_1)k - p_1q_1 - p_2q_2 - p_3q_3 \\
[q,p] &= qp - pq = 2(q_2p_3 - q_3p_2)i + 2(q_3p_1 - q_1p_3)j + 2(q_1p_2 - q_2p_1)k \\
[p,q] &= pq - qp = -(qp - pq) = -[q,p]
\end{aligned}$$

Note that the bracket of two vectors is a vector. We can add and subtract and scale vectors, and now we have a way to multiply them and stay within the space of vectors (vector-vector multiplication will yield a non-pure quaternion). We say that the vectors form an algebra because we have both well-defined addition and multiplication on them. The vectors are called skew-commutative with respect to the bracket because $[q,p] = -[p,q]$ (this may seem trivial, but it isn't always). Now consider

$$\begin{aligned}
[p,[q,r]] + [q,[r,p]] + [r,[p,q]] &= [p, qr - rq] + [q, rp - pr] + [r, pq - qp] \\
&= (pqr - qrp) + (qrp - rpq) + (rpq - pqr) \\
&= 0
\end{aligned}$$

$[p,[q,r]] + [q,[r,p]] + [r,[p,q]] = 0$ is called the Jacobi identity.

Any algebra that is skew-commutative (with respect to the bracket) and satisfies the Jacobi identity is called a Lie algebra.

Someone who has studied Lie groups will probably not be surprised that the vector form a Lie algebra. Note that the space of vectors is orthogonal to 1, the multiplicative identity element of the unit quaternions. Since the unit quaternions form a Lie group, the tangent space at the (multiplicative) group unit is a Lie algebra; this is true for every Lie group. Every quaternion in the tangent space at 1 has the form $1 + u$, where u is a vector. Hence, this tangent space (where we treat the bracket as the multiplication operation) is isomorphic to the space of vectors (where the bracket is the multiplication operation). We can conclude that the space of vectors is a Lie algebra, which he have already verified independently.

We could have found directly that the vectors form a Lie algebra by computing the bracket of elements of $\mathbb{R}^3$. Recall that the cross product on $\mathbb{R}^3$ maps each pair of elements of $\mathbb{R}^3$ back into $\mathbb{R}^3$, and that the cross-product is skew-commutative ($\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$).

$$\begin{aligned}
[\mathbf{x}, \mathbf{y}] &= \mathbf{x} \times \mathbf{y} - \mathbf{y} \times \mathbf{x} \\
&= 2\mathbf{x} \times \mathbf{y}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{y}, \mathbf{x}] &= \mathbf{y} \times \mathbf{x} - \mathbf{x} \times \mathbf{y} \\
&= 2\mathbf{y} \times \mathbf{x} \\
&= -2[\mathbf{x}, \mathbf{y}],
\end{aligned}$$

so $\mathbb{R}^3$ is skew-commutative with respect to the bracket. We can also show that the elements of $\mathbb{R}^3$ satisfy the Jacobi identity.

Since there is an isomorphism between the vectors (where multiplication is considered the bracket) and the matrices L_u where u is a vector, those matrices also form a Lie algebra. The same is true of the matrices R_u, with u a vector, for the same reason. These sets are called matrix Lie algebras.